Theory of Aquifer Tests

U.S. GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1536-E

The analysis of steady-state test data for a leaky aquifer can thus be summarized in the following three simple procedures:

1. Select for plotting only the drawdown data which are within the region where drawdowns have levelled off.
2. Use equations 36, 37, and 42 with a logarithmic plot of s versus r, matched to the leaky-aquifer type curve (fig. 27), only if the observed data and resulting computations produce values of x greater than 0.03.
3. Use equations 2, 4, and 43a with a semilogarithmic plot of s versus log r if the data and resulting computations produce values of x less than 0.03.

The earliest observations of drawdown in each observation well, when s is small, should conform to the Theis nonequilibrium type curve for the infinite (nonleaky) aquifer if the rate of leakage from the confining bed is comparatively small. The coefficient of storage for the artesian aquifer can be determined under these conditions from the earliest observations of drawdown (Jacob, 1946, p. 204). The computed coefficient of transmissibility should be checked by comparing the value obtained from matching the earliest data to the nonequilibrium type curve with the value obtained by matching the later data to the steady-state leaky-aquifer type curve. If consistency of the T values is not obtained, then the leakage may be causing too much deviation at the smaller values of t to permit application of the Theis nonequilibrium formula.

VARIABLE DISCHARGE WITHOUT VERTICAL LEAKAGE

By R. W. Stallman

CONTINUOUSLY VARYING DISCHARGE

The rate at which water is pumped from a well or well field commonly varies with time in response to seasonal changes in demand. For instance, the pumping rate, as shown by records of daily or monthly discharge, is often found to be varying continuously. Where this element of variability is recognized in ground-water problems, the analytical methods that are described in the preceding sections of this report are not applicable without some modification or approximation. Exact equations could perhaps be developed for the case of continuously varying discharge, but the cost of analysis, in terms of time and effort, would likely be prohibitive considering that a separate and specific solution would be required for each problem. It is considered more expedient, therefore, to utilize the existing analytical methods, rendering them applicable to the field situation by introducing tolerable approximations of the field conditions. As an example, consider a situation where the pumping rate in a well (which may also represent a well field) tapping an artesian aquifer varies continuously with time in the manner indicated by the smooth curve shown in

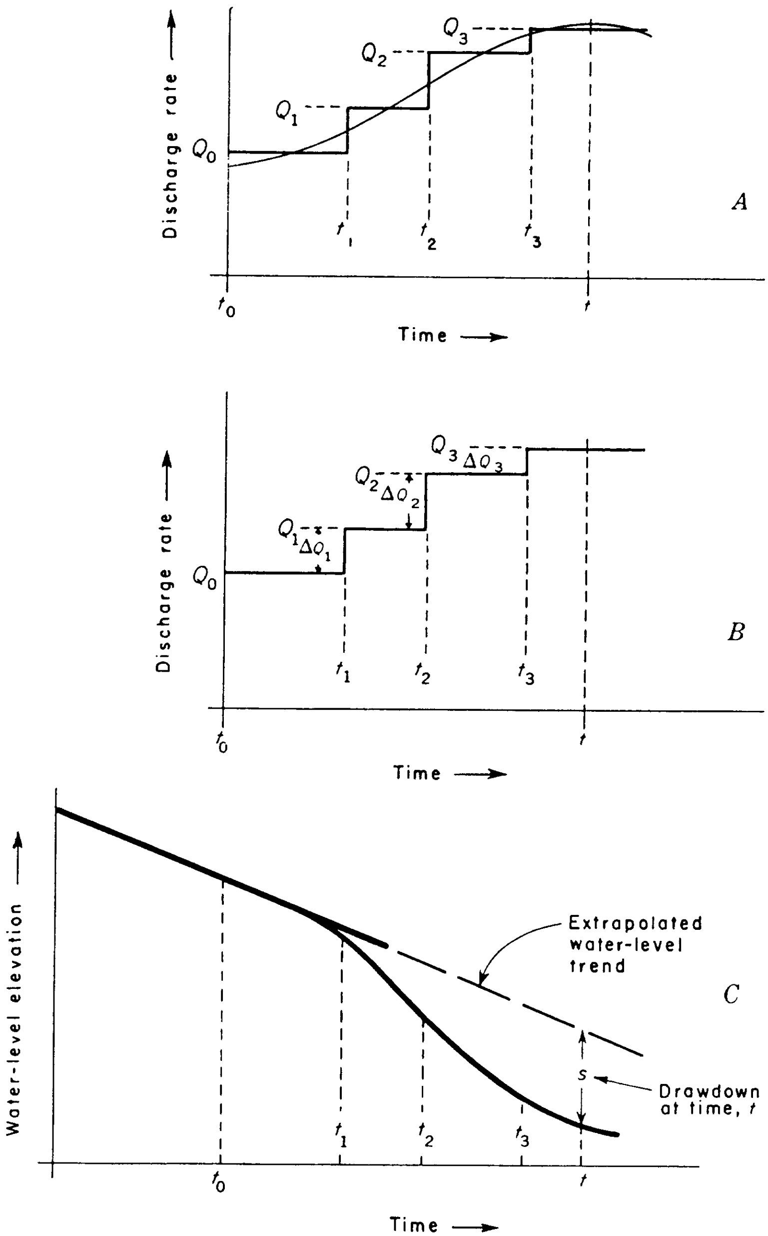


FIGURE 28.—Nomenclature for continuously varying discharge.

figure 28*A*. This smooth curve may be approximated by the series of steps shown, and the analysis of each step may be undertaken starting with conventional theory and equations. Thus the Theis nonequilibrium formula (eq. 6) can be used to construct a type curve for analyzing the observed drawdowns caused by the stepped pumping rates indicated in figure 28*A* and *B*. The drawdown, s, at distance r from the pumped well, at any time t, is

$$s=s_1+s_2+s_3 \; . \; . \; . \; . \; . \; . \; . +s_n \tag{44}$$

where the subscripts refer to the ΔQ values of figure 28*B*. The zero reference time t_0 is chosen arbitrarily so that the effects of the antecedent rate of pumping, Q_0, are established as a regular trend that can be projected or extrapolated with certainty, as shown in figure 28*C*, over the time span occupied by the stepped pumping rates. Applying the Theis nonequilibrium formula to define each of the drawdown components given in equation 44, there follows,

$$s=\frac{114.6}{T}\,[\Delta Q_1W(u)_1+\Delta Q_2W(u)_2+\Delta Q_3W(u)_3 \; . \; . \; . \; . \; . +\Delta Q_nW(u)_n]. \tag{45}$$

The corresponding u values are

$$u_1=\frac{1.87r^2S}{T(t-t_1)};\; u_2=\frac{1.87r^2S}{T(t-t_2)};\; u_3=\frac{1.87r^2S}{T(t-t_3)};\; . \; . \; . \; ;\; u_n=\frac{1.87r^2S}{T(t-t_n)}. \tag{46}$$

Therefore,

$$u_2=u_1\left[\frac{t-t_1}{t-t_2}\right];\; u_3=u_1\left[\frac{t-t_1}{t-t_3}\right];\; . \; . \; . \; . \; . \; ;\; u_n=u_1\left[\frac{t-t_1}{t-t_n}\right]. \tag{47}$$

Inspection of equations 46 and 47 should indicate that virtually an infinite number of type curves can be constructed for solving equation 45. For practical purposes, however, only a family of curves need be constructed.

It can be seen from equations 46 that the relation between the u values is dependent on the value of t selected. For any given value of t, the values of u are proportional to the constant $1.87r^2S/T$. Therefore the family of curves must be constructed using t and $1.87\,r^2S/T$ as independent variables and $\sum_1^n \Delta QW(u)$ as the dependent variable. This is accomplished by first assuming several values of $1.87\,r^2S/T$ for a particular value of t. Values of u_1 are then computed for that t for the assumed values of $1.87r^2S/T$ using the first of equations 46. Equations 47 are then used to compute values of u_2, u_3 . . . u_n for each assumed value of $1.87r^2S/T$. These in turn determine (see table 2) the corresponding $W(u)$ values, which are used to compute the quantity in brackets (the sum of all the $\Delta QW(u)$ terms) in equation 45. Thus a set of values is produced for the sum of the $\Delta QW(u)$ terms, corresponding with the assumed values of 1.87 r^2S/T and all are related to one assumed value of t. This computing procedure is repeated for each value of t in a whole set of t values selected to span a time range that will permit drawing the family of type curves, shown schematically in figure 29, through the same time

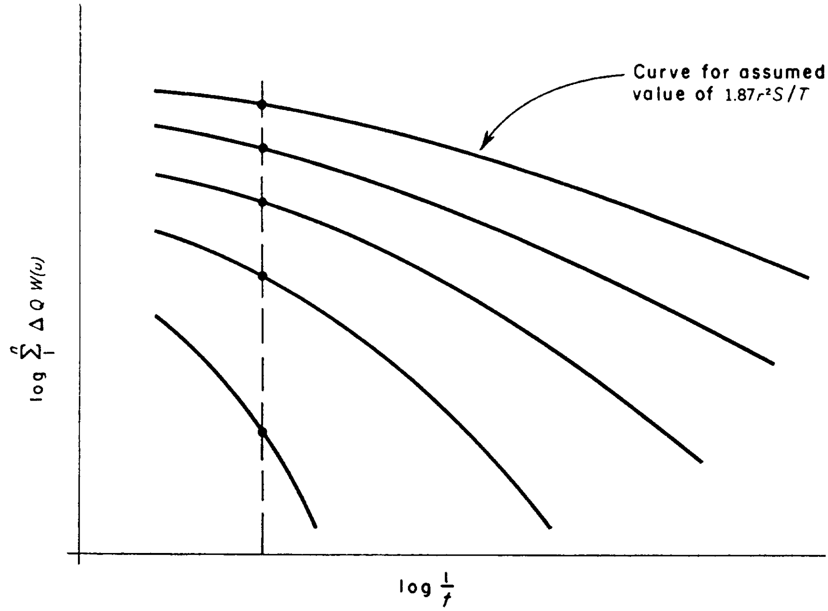


FIGURE 29.—Schematic plot of family of type curves for problems involving continuously varying discharge.

interval covered by the drawdown observations in the aquifer. The field-data plot of log s versus log $1/t$ is superposed on the family of type curves, taking care that the logarithmic time scales of the two graphs are exactly matched. The data plot is then shifted along the $\sum_1^n \Delta QW(u)$ axis until the position is found where the curvature of the data plot is identical with an underlying type curve or with an interpolated type curve position. It follows that this serves to identify the data curve with a specific value of $1.87r^2S/T$. Values of s and $\sum_1^n \Delta QW(u)$ are read from a point common to both graphs and entered in equation 45 to solve for T. The computed value of T can then be used with the value of $1.87r^2S/T$ to solve for S. Should several observation wells at different radii be available, it may be more convenient to construct a type curve suitable for matching with the observed drawdown profile. For a selected observation time, values of $\sum_1^n \Delta QW(u)$ and $1.87r^2S/T$ are taken from figure 29 and used to construct a new type curve by plotting $\log \sum_1^n \Delta QW(u)$ against $\log (1.87 r^2S/T)$. This new logarithmic type curve, drawn for a selected time, t, can be matched with a logarithmic data plot of s versus r^2, drawn for the same time t.

INTERMITTENT OR CYCLIC DISCHARGE

Analysis of drawdown data by means of the methods described in the preceding section is likely to require a large amount of calculation. However, for certain specific kinds of discharge variations the analysis can be simplified considerably. The detailed solutions of two specific cases have been described by Theis and Brown (1954). One of the problems solved was that of computing the drawdown occurring in a well being operated in a regular cycle of pumping at a constant rate for a given time interval, then resting for a given time interval. Their final equation, in the usual Survey units, for drawdown in the pumped well after n cycles of operation is

$$s_n=\frac{264Q}{T}\log_{10}\frac{1\cdot 2\cdot 3\cdot \ldots\ldots n}{(1-p)(2-p)(3-p)\ldots\ldots(n-p)}, \qquad (48)$$

where p is the fractional part of the cycle during which the well is pumped. In part, the simple form of equation 48 was obtained by utilizing the semilog approximation (eq. 4) of the Theis nonequilibrium formula. Many regular operational cycles are easily generalized and analysis may lead to a final expression comparable, in simplicity, to equation 48.

CHANNEL METHODS—LINE SINK OR LINE SOURCE

CONSTANT DISCHARGE

NONSTEADY STATE, NO RECHARGE

As early as 1938 Theis (Wenzel and Sand, 1942, p. 45) had developed a formula for determining the decline in artesian head at any distance from a drain discharging water at a uniform rate. In 1949 Ferris (1950) derived a formula that can be shown to be identical with the one derived by Theis. The development is based on the following assumptions: (a) the aquifer is homogeneous, isotropic, and of semi-infinite (bounded on one side only by the stream) areal extent; (b) the discharging drain completely penetrates the aquifer; (c) the aquifer is bounded by impermeable strata above and below; (d) the flow is laminar and unidimensional; (e) the release of water from storage is instantaneous and in proportion to the decline in head; and (f) the drain discharges water at a constant rate.

Slightly modifying the form used by Ferris, the drain formula can be written nondimensionally, as

$$s=\frac{Q_b x}{2T}\left[\frac{e^{-u^2}}{u\sqrt{\pi}}-1+\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Tt/S}}} e^{-u^2}du\right], \qquad (49)$$

where

$$u=x\sqrt{\frac{S}{4Tt}},$$

or

$$u^2=\frac{x^2S}{4Tt}. \qquad (50)$$

Ferris suggested that the quantity in brackets be written symbolically, for convenience, as $D(u)$ which is to be read "drain function of u," and in this report the subscript q will identify it with the constant discharge situation. Equations 49 and 50 can therefore be rewritten in abbreviated form, in the usual Survey units, as

$$s=\frac{720Q_b x}{T}D(u)_q \qquad (51)$$

and

$$u^2=\frac{1.87x^2S}{Tt}, \qquad (52)$$

where

s=drawdown, in feet, at any point in the vicinity of the drain discharging at a constant rate,

Q_b=constant discharge (that is, base flow) of the drain, in gallons per minute per lineal foot of drain,

x=distance, in feet, from the drain to the point of observation,

t=time, in days, since the drain began discharging,

and S and T have the meaning and units already defined.

From inspection of equations 51 and 52 it follows that if s can be measured at several values of t, and if x and Q_b are known, then S and T can be determined. However, the occurrence of two unknowns and the nature of the drain function make an exact analytical solution impossible and trial solution most laborious. A graphical solution of superposition, similar to the one devised by Theis for solution of his nonequilibrium formula, affords a simple solution of equation 51.

The first step in constructing the type curve is to assume values of u and compute the corresponding values for $D(u)_q$ from equation 49, which can be done easily with the aid of published tables (U.S. Natl. Bur. of Standards, 1954). Values of $D(u)_q$ and u^2 for values of u from 0.0510 to 1.0000 are given in table 5. These data are then used

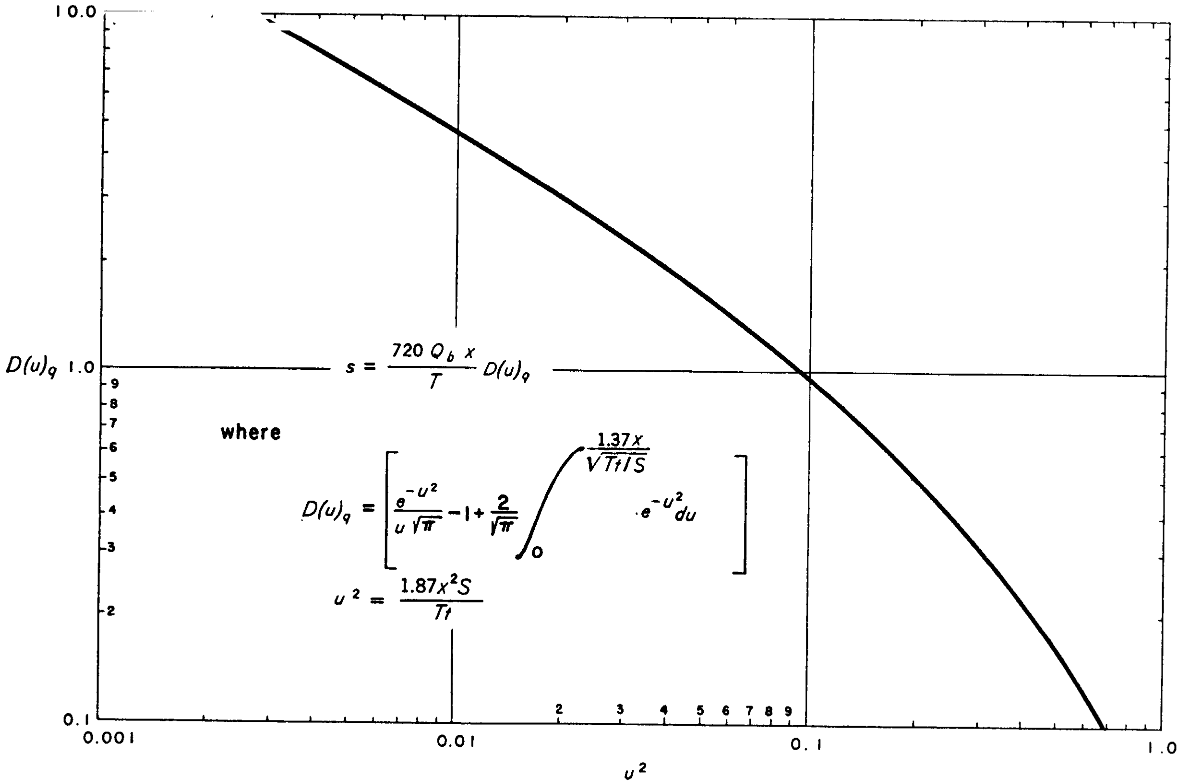


Figure 30.—Logarithmic graph of the drain function $D(u)_q$ for channel method—constant discharge.

to prepare a type curve on logarithmic coordinate paper by plotting values of u or u^2 against values of $D(u)_q$. Such a type curve is shown in figure 30 and, for convenience in subsequent computation, values of $D(u)_q$ have been plotted against values of u^2.

Rearranging equations 51 and 52 and taking the log of both sides, there follows:

$$\log s = \left[\log \frac{720\, Q_b x}{T}\right] + \log D(u)_q, \qquad (53)$$

and

$$\log \frac{x^2}{t} = \left[\log \frac{T}{1.87\, S}\right] + \log u^2. \qquad (54)$$

For a given test, the bracketed parts of equations 53 and 54 are constant and log $D(u)_q$ is related to log u^2 in the manner that log s is related to log (x^2/t). Therefore, if values of the drawdown, s, are plotted versus x^2/t on logarithmic tracing paper having the same log scale as the type curve for the drain formula, the curve of observed data will be similar to the type curve. The data curve may thus be superposed on the type curve, with the coordinate axes held parallel, and translated to the position where the observed data coincide or make the best fit with the type curve. When this matching position has been found, an arbitrary point is selected, common to both curves, and the coordinates of this common point are used to solve equations 53 and 54 for T and S.

TABLE 5.—*Values of* $D(u)_q$, u, *and* u^2 *for channel method—constant discharge formula.*[a]

[Data for plotting type curve (fig. 30) for equation 51. After Ferris (1950)]

u	u^2	$D(u)_q$	u	u^2	$D(u)_q$
0.0510	0.0026	10.091	.2646	.070	1.280
.0600	.0036	8.437	.3000	.090	1.047
.0700	.0049	7.099	.3317	.110	.8847
.0800	.0064	6.097	.3605	.130	.7641
.0900	.0081	5.319	.4000	.160	.6303
.1000	.010	4.698	.4359	.190	.5327
.1140	.013	4.013	.4796	.230	.4370
.1265	.016	3.531	.5291	.280	.3516
.1414	.020	3.069	.5745	.330	.2895
.1581	.025	2.657	.6164	.380	.2426
.1732	.030	2.355	.6633	.440	.1996
.1871	.035	2.120	.7071	.500	.1666
.2000	.040	1.933	.7616	.580	.1333
.2236	.050	1.648	.8124	.660	.1084
.2449	.060	1.440	.8718	.760	.08503
			.9487	.900	.06207
			1.0000	1.000	.05026

Despite the restrictive assumptions upon which it is based, the drain formula, as it has been called, has been applied successfully in determining the coefficients of transmissibility and storage of an aquifer and in estimating the pickup by or leakage from drains.

Discussion and comparison of various ways of plotting the type curves for the drain function $D(u)_q$ and the well function $W(u)$ (see figures 30 and 23) are given by Warren (1952, written communication).

CONSTANT HEAD

NONSTEADY STATE, NO RECHARGE

By R. W. STALLMAN

The decline in artesian head at any distance from a stream or drain, whose course may be approximated by an infinite straight line, subsequent to a sudden change in stream stage, can be found by borrowing the solution to an analogous heat-flow problem (Ingersoll, Zobel, and Ingersoll, 1948, p. 88). It is assumed that (a) the stream occurs along an infinite straight line and fully penetrates the artesian aquifer; (b) the aquifer is semi-infinite in extent (bounded on one side only by the stream); (c) the head in the stream is abruptly changed from zero to s_0 at time $t=0$; (d) the direction of ground-water flow is perpendicular to the direction of the stream; and (e) the change in the rate of discharge from the aquifer is derived from changes in storage by drainage after $t=0$. Substituting ground-water nomenclature in the heat-flow equation, the distribution of drawdown in the artesian aquifer is found to be

$$s_0-s=\frac{2s_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Tt/S}}} e^{-u^2}du,$$

or

$$s=s_0\left[1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Tt/S}}} e^{-u^2}du\right]=s_0D(u)_h, \tag{55}$$

where $D(u)_h$ replaces the quantity in brackets and represents the drain function of u for the constant head situation, and where

$$u^2=\frac{x^2S}{4Tt}. \tag{56}$$

In the foregoing expressions x is the distance from the stream or drain to the point at which the decline in artesian head, s, is observed or known, and s_0 is the abrupt change in stream stage at $t=0$. Other symbols are as previously defined. (Note: In equation 55 the integral expression and its coefficient constitute what mathematicians have labelled the error function, written as "erf." The bracketed portion of equation 55 is identified as the complementary error function, written as "cerf".)

The relation expressing the discharge from the aquifer, per unit length of stream channel, Q_b, resulting from the change in stream stage, can also be found in texts on heat flow (Ingersoll, Zobel, and Ingersoll, 1948, p. 90). When written using ground-water notation, and multiplying by 2 to account for the water contributed from both sides of the stream the equation has the form

$$Q_b=\frac{2s_0}{\sqrt{\pi t}}\sqrt{ST}. \tag{57}$$

Equations 56 and 57 now afford a means for evaluating the two unknowns T and S, inasmuch as the ratio $S:T$ is determined from equation 56 and the product ST is obtained from equation 57.

Comparing equations 55 and 56 the use of the method of superposed graphs, described in previous sections, is again indicated as the most logical means of solution because log s evidently varies with log x^2/t in the same manner that log $D(u)_h$ varies with log u^2. Thus the solution of equation 56 for the ratio $S:T$ will evidently require matching a logarithmic data plot of values of s versus corresponding values of x^2/t (or simply $1/t$ if only one observation well is available) to a logarithmic type curve prepared by plotting values of $D(u)_h$ versus corresponding values of u^2. Such a type curve is shown in figure 31, prepared from the drain function values given in table 6.

If equations 56 and 57 are rewritten using the usual Survey units (except for Q_b which is the base flow in gallons per minute per foot of stream length), they become

$$u^2=\frac{1.87x^2S}{Tt}, \tag{58}$$

and

$$Q_b=2.15\times10^{-3}s_0\sqrt{\frac{ST}{t}}. \tag{59}$$

TABLE 6.—*Values of* $D(u)_h$, u, *and* u^2 *for channel method—constant head formula*

[Data for plotting type curve (fig. 31) for equation 55. Prepared by R. W. Stallman. Values of $D(u)_h$, for selected values of u^2 or u, were computed with the aid of U.S. Natl. Bureau of Standards tables (1954)].

u	u^2	$D(u)_h$	u	u^2	$D(u)_h$
0.03162	0.0010	0.9643	0.6325	0.40	0.3711
.04000	.0016	.9549	.7746	.60	.2733
.05000	.0025	.9436	.8944	.80	.2059
.06325	.0040	.9287	1.000	1.00	.1573
.07746	.0060	.9128	1.140	1.30	.1069
.08944	.0080	.8994	1.265	1.60	.0736
.1000	.010	.8875	1.378	1.90	.0513
.1265	.016	.8580	1.483	2.20	.0359
.1581	.025	.8231	1.581	2.50	.0254
.2000	.040	.7730	1.643	2.70	.0202
.2449	.060	.7291	1.732	3.00	.0143
.2828	.080	.6892	1.789	3.20	.0114
.3162	.10	.6548			
.4000	.16	.5716			
.5000	.25	.4795			

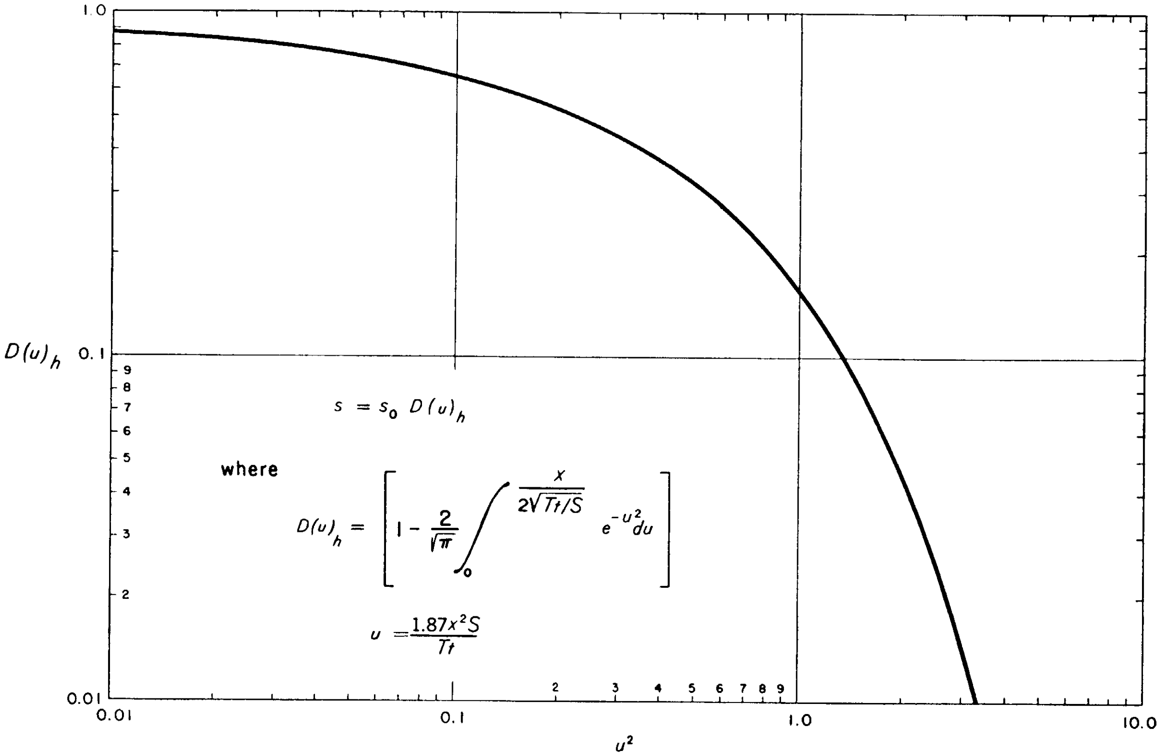


Figure 31.—Logarithmic graph of the drain function $D(u)_h$ for channel method—constant head.

Equations 55 and 57 define the changes in head and flow which occur in the aquifer if the stream stage is abruptly changed. Therefore the stipulation of no recharge implies only that the rate of recharge must be constant for a sufficient interval of time, so that the regional water-level trends can be extrapolated with accuracy throughout the period in which the changes in the aquifer are being observed.

Water-level data from wells near the stream may, in some field situations, yield erratic results, depending on the flow pattern in the aquifer in the vicinity of the stream. For example, if the stream only partly penetrates the aquifer the flow lines in the aquifer will obviously bend upward as they approach the stream, thereby producing vertical components of flow. Thus the smaller the distance x, between observation well and stream, the greater the errors inherent in the observed water levels. This in turn means that as x decreases, the error in the computed value of S/T increases. It should also be realized that instantaneous or abrupt lowering of stream stage is seldom possible, which means that the determination of a reference or zero time is difficult. Thus observations made a short time after the stream stage is lowered may be somewhat unreliable. In general, therefore, it would seem prudent to favor the data collected at comparatively large values of x and large values of t to provide the most reliable basis for analysis.

Where it is known that the stream or drain penetrates only a part of the aquifer thickness, the following adjustment procedure, though not yet proven by field trial, may offer a means for determining more realistic values for T and S. It should be evident that in a field situation of this kind, the change in head in the stream channel is not as effective in producing head changes throughout the aquifer as when the stream is fully penetrating. Near the stream, ground-water levels adjust quickly to changes in stream stage, but part of the adjustment is caused by the bending of the flow lines. It can be assumed, however, that at some relatively short distance x_0 away from the stream the bending of the flow lines in the aquifer will be small enough so that the effects on the head values may be neglected. Thus, for distances greater than x_0 the flow lines may be considered parallel—that is, flow is essentially one-dimensional. The change in head in the aquifer, at the distance x_0, may therefore be considered as an effective value of s_0 and it is related to the changes in head throughout the aquifer, for all distances greater than x_0, in the manner described by equations 55 and 56.

In effect this reasoning means that the real partly penetrating stream, in which the stage was abruptly changed an amount s_0, is being replaced (at the distance x_0) by a theoretical fully penetrating stream in which the stage change may be regarded as essentially

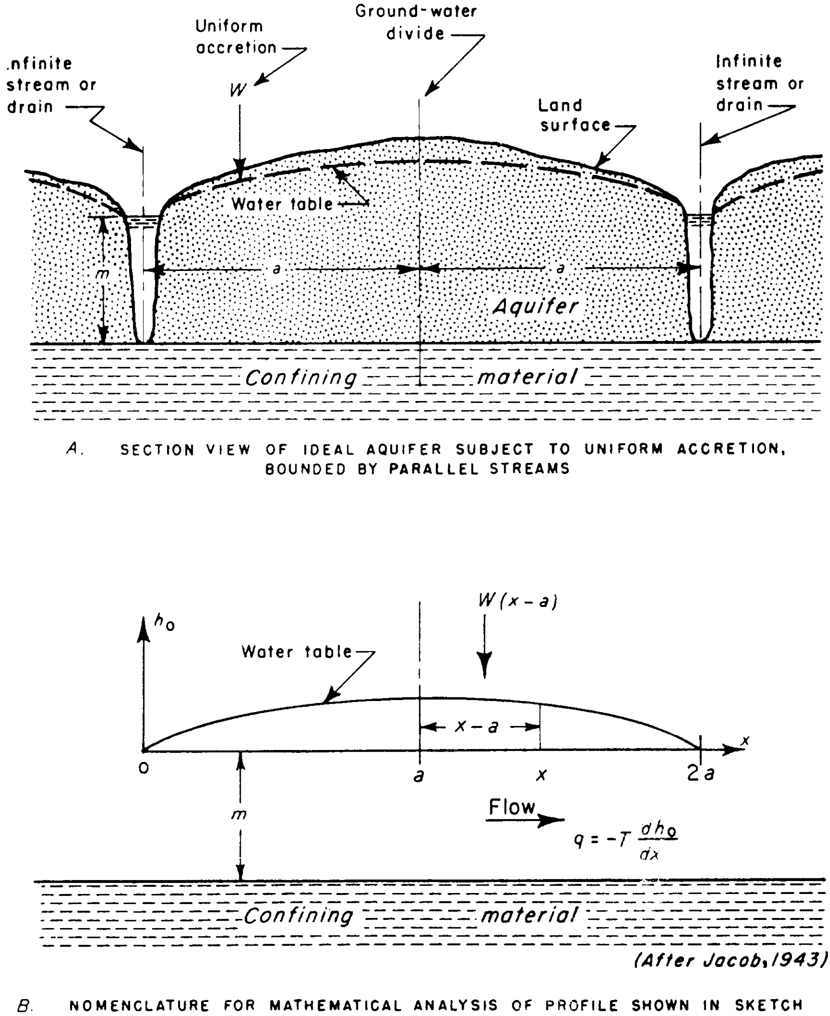


FIGURE 32.—Section views for analyzing steady-state flow in hypothetical aquifer of large thickness with uniform accretion from precipitation.

abrupt but of a lesser magnitude which shall be termed an effective value of s_0.

This socalled effective value of s_0 can be computed from equation 55 after superposing the data and type curves in the manner already described. The critical distance x_0 can also be computed, using the coordinates for a point common to the matched data and type curves, if more than one observation well is available. For each observation well the ratio S/T is computed, using the distance from the well to the real stream channel as a first estimate in equation 56. If the S/T values thus determined are not alike, the equation for u^2 is adjusted to read as follows:

$$u^2=\frac{1.87(x-x_0)^2S}{Tt}.$$

An estimate of x_0 is then made, and S/T values for each observation well are recomputed using the effective distance to the stream $(x=x_0)$. If the data and field conditions are sufficiently ideal to permit an accurate analysis, several assumed values of x_0 will indicate the one that will produce the closest agreement in the computed values of S/T.

It is pointed out that it is difficult to assess the true value of these adjustment procedures, inasmuch as the opportunity for applying them to a specific field problem has not yet been afforded.

STEADY STATE, UNIFORM RECHARGE

A problem of considerable practical interest is that of estimating the base flow of streams, or the effective average rate of ground-water recharge, from the shape of the water table. Consider the case of an aquifer bounded on two sides by fully penetrating parallel streams of infinite length as shown in figure 32*A*. It is assumed that the aquifer is homogeneous and isotropic, and that the aquifer is recharged at a rate of accretion, W, that is constant with respect to time and space. Flow is therefore one-dimensional and a ground-water divide is created at distance a, midway between the streams (see figure 32*B*). Jacob (1943, p. 566) has given the equation of steady-state profile as

$$h_0=\left(\frac{a^2W}{2T}\right)\left(\frac{2x}{a}-\frac{x^2}{a^2}\right),$$

or

$$\frac{T}{W}=\frac{ax}{h_0}-\frac{x^2}{2h_0}, \tag{60}$$

where

W=constant rate of recharge to the water table;
a=distance from the stream to the ground-water divide;
x=distance from the stream to an observation well;
h_0=elevation of the water table, at the observation well, with respect to the mean stream level.

It is frequently convenient to express the rate of recharge, W, in inches per year, while a, x, and h_0 are expressed in feet, and T is in the usual Survey units. Equation 60 is then rewritten in the form

$$T=1.71(10^{-3})W\left(\frac{ax}{h_0}-\frac{x^2}{2h_0}\right). \tag{61}$$

In the absence of artificial withdrawal of water from an aquifer, the net recharge must equal the natural discharge, provided changes in storage are insignificant. Although it is recognized that under natural conditions there are variations of W in time, it should be apparent that if average or mean ground-water levels are used in equation 61, a figure for W equivalent to an effective average accretion rate will result.

Equation 61 can be solved if the average (in time) contribution, Q_b, to the base flow of the stream, per unit length of channel can be determined from stream-flow measurements. If Q_b is expressed in gallons per minute per foot of stream channel, then

$$W=8.44(10^5)\frac{Q_b}{2a},$$

or

$$W=4.22(10^5)\frac{Q_b}{a}. \tag{62}$$

Equation 62 permits determination of W which can be used in equation 61 in computing the value of T.

In many field situations, of the type postulated here, general appraisal of the occurrence of ground water may indicate that the ground-water divide is parallel to the stream course, although the distance a will be unknown. If two observation wells are available, it is possible to compute a value for a and a value for the ratio $586T/W$. If a larger number of observation wells is available, a graphical solution may be used to solve for the values of a and $586T/W$. The procedure requires observation of the distances, x, from the stream to the individual observation wells and the corresponding values of h_0. Using the data from one observation well, and arbitrarily selecting several values of a, the corresponding values of $586T/W$ are computed after appropriate substitution in equation 61. The computed values of $586T/W$ are plotted against the corresponding assumed values of a and a smooth curve is drawn through the plotted points. In similar fashion a curve is drawn on the same graph for each observation well. The coordinates of the single point at which all the curves intersect give the particular values of a and of $586T/W$ which will satisfy all the available data.

SINUSOIDAL HEAD FLUCTUATIONS

Werner and Noren (1951) and later Ferris (1951) independently analyzed the problem of fluctuations of water levels in wells in response to sinusoidal changes in stage of nearby surface-water bodies. The solution to this general type of problem has long been available in other fields of science and as Ferris indicates it may conveniently be found in reference works on heat flow such as the text by Ingersoll, Zobel, and Ingersoll (1948, p. 46–47). Translated into ground-water terms, the solution requires an isotropic semi-infinite artesian aquifer of uniform thickness in full contact, along its one boundary, with a surface-water body that may be considered an infinite line source. Within the aquifer the change in water storage is assumed to occur instantaneously with, and at a rate proportional to, the change in pressure. Ferris (1951, p. 3) shows that the equation for the range of ground-water fluctuation in an observation well, a distance x from the aquifer contact with the surface-water body, whose stage is changing sinusoidally, has the nondimensional form

$$s_r = 2s_0 e^{-x\sqrt{\pi S/t_0 T}} \tag{63}$$

or, in the usual Survey units,

$$s_r = 2s_0\, e^{-4.8x\sqrt{S/t_0 T}}, \tag{64}$$

where

s_r=range of ground-water stage, in feet;
s_0=amplitude or half range of the surface-water stage, in feet;
x =distance from the observation well to the surface-water contact with the aquifer ("suboutcrop"), in feet;
t_0=period of the stage fluctuation, in days;
S=coefficient of storage; and
T=coefficient of transmissibility, in gallons per day, per foot.

For convenience equation 64 can be written

$$2.1\sqrt{S/t_0 T} = \frac{-\log_{10}(s_r/2s_0)}{x}. \tag{65}$$

The form of equation 65 suggests a semilogarithmic plot of the log of the range ratio, $s_r/2s_0$, for each observation well, versus the corresponding distance, x, as an expedient method of analyzing the observed data. The right-hand member of equation 65 represents the slope of the straight line that should be defined by the plotted data. If the change in the logarithm of the range ratio is selected over one log cycle of the plot, equation 65 becomes

$$2.1\sqrt{S/t_0 T} = -\frac{1}{\Delta x},$$

or

$$T = \frac{4.4(\Delta x)^2 S}{t_0}. \tag{66}$$

If the field conditions fulfill the assumptions made in deriving equation

66, the straight line drawn through the data on the semilogarithmic plot should pass through the origin of the coordinate axes—that is, should intercept a value of $r/2s_0=1$ at a value of $x=0$. For the case of a stream of substantial width, partially penetrating the artesian aquifer, this intercept will usually be found at negative values of x, indicating an "effective" distance offshore to the suboutcrop. This effective distance may or may not have physical significance depending on the nature of the flow field in the vicinity of the stream. For example, if the stream does not cut through the upper confining bed, stage changes in the surface-water body may still provide, through the changes in loading that are involved, a source of sinusoidal head fluctuation in the artesian aquifer along the stream location. Unless the loading changes are completely effective (100 percent tidal efficiency) in producing a stage range of $2s_0$ at $x=0$ in the aquifer, the straight-line intercept with the x axis will again be at some negative value of x, and will in part be an indication of the efficiency with which the aquifer skeleton accepts the changes in loading. However, it should be observed that regardless of the exact value of s_0 in the aquifer, at $x=0$, the slope of the data plot described is unaffected. Therefore the T value computed by means of equation 66 will be correct regardless of the actual value in the aquifer of s_0.

It is seen from equation 66 that it is necessary to know S in order to solve for T. Frequently, when the coefficient of storage is not known, it is possible to make a reasonable estimate of its value by studying the well logs and water-level records.

The lag in time of occurrence of a given maximum or minimum ground-water stage, following the occurrence of a similar surface-water stage, is given by Ferris (1951) as

$$t_1=\frac{x}{2}\sqrt{\frac{t_0S}{\pi T}}, \tag{67}$$

where t_1 is the lag in time. Solving this equation for the coefficient of transmissibility, and rewriting in terms of the usual Survey units, there follows

$$T=0.60t_0S\left(\frac{x}{t_1}\right)^2. \tag{68}$$

The only variables in equation 68 are evidently x and t_1. Thus an arithmetic plot is suggested with the value of the distance, x, for each observation well, plotted against the corresponding value of the time lag, t. The slope of the straight line that should be defined by these plotted points will then give the value of x/t_1, which appears to the second power in equation 68. If the straight line that is drawn through the plotted points should intersect the zero–timelag axis at a negative value of x it may be an indication of the effective distance offshore to the suboutcrop.

In those situations where the aquifer is not fully penetrated or where it is under water-table (unconfined) conditions, the methods of analysis described in this section will be satisfactory if (a) the observation wells are far enough from the suboutcrop so that they are unaffected by vertical components of flow and (b) the range in fluctuations is only a small fraction of the saturated thickness of the formation.

AREAL METHODS

NUMERICAL ANALYSIS

By R. W. Stallman

The equations presented in the preceding sections of this paper were derived by means of the calculus. Darcy's law combined with the equation of continuity (Rouse, 1950, p. 326) yielded the basic differential equations that described states of flow. In turn, solutions to these differential equations were found that satisfy the boundary conditions of a particular problem. Certain generalizations were made in regard to the boundary conditions so as to provide rather specific equations, which can be used with convenience to obtain a solution to the field problem. Among these generalizations are the assumptions of a constant head or discharge at some point or line, homogeneity of the aquifer, simple geometric form or shape of the aquifer, and complete penetration of the well or stream. Certainly for many field problems these conditions are fulfilled to a sufficient degree that the available formulas can be used to obtain reliable approximations of the values of T, S, or W. However, the ground-water hydrologist frequently encounters problems for which the complicated boundary conditions cannot be expressed by simple mathematical relations. Furthermore, the complicated boundary conditions related to a given field problem seldom recur in nature in the same combination. Thus it could be poor economy to spend a large amount of time and energy deriving complicated analytical equations whose application might be limited to one problem. Under such circumstances it may be found more expedient to use numerical methods of analysis for the quantitative investigation. Numerical methods have been used in other sciences for some time for the same purpose. Basic formulas and procedures have been described by Southwell (1946, 1940) and many of his colleagues (Shaw, 1953). Scarborough (1950) and Milne (1953) have written extensively on the same subject and an application of numerical methods to ground-water investigations has been described by Stallman (1956).

The formal derivation of analytical equations for describing ground-water movement involves application of the rules of calculus for integrating, or summing, an infinite set of infinitesimal changes in head between two or more points in the flow region under study. In lieu of application of the rules of calculus to perform this addition conveniently, it would be possible to accomplish the same thing simply by addition of the infinite set of infinitesimals. The latter method is, of course, impractical unless approximations are introduced. Thus an area may be calculated by considering that it is composed of small but finite parts, each having an area $(\Delta x)(\Delta y)$. By means of this more coarse subdivision of the area, the problem is reduced to the addition of a sensibly small and finite set of component parts instead of the infinite set of infinitesimal areas $(dx)(dy)$ postulated by the orthodox calculus. In brief this constitutes the basis of numerical analysis as applied in finding solutions to differential equations: the substitution of finite entities for the differential forms that appear in the fundamental differential equations.

Consider, for example, the differential equation describing two-dimensional unsteady flow in a homogeneous and isotropic aquifer, subject to recharge at a steady rate of accretion W. It can be shown that this equation has the form

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\left(\frac{S}{T}\right)\frac{\partial h}{\partial t}-\frac{W}{T}, \tag{69}$$

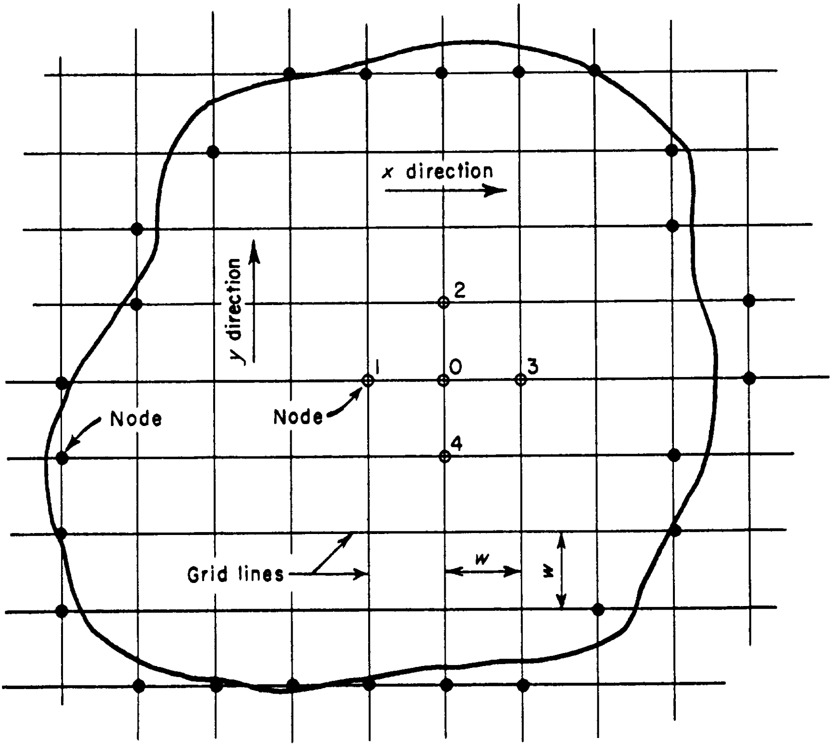
where h is the head at any point whose coordinates are x, y. Let the differential lengths dx and dy be expanded so that each can be considered equivalent to a finite length w, and similarly let dt be considered equivalent to Δt. A plan representation of the region of flow to be studied may then be subdivided by two systems of equally spaced parallel lines at right angles to each other. One system is oriented in the x direction and the other in the y direction and the spacing of the lines equals the distance w (see fig. 33). A set of 5 gridline intersections or nodes, selected in the manner shown on figure 33, is called an array. Accordng to Southwell (1946, p. 19) the first two differentials in equation 69 can be expressed, in terms of the head values at the nodes in the array, in the following fashion:

$$\frac{\partial^2 h}{\partial x^2}\approx\frac{h_1+h_3-2h_0}{w^2}$$

and

$$\frac{\partial^2 h}{\partial y^2}\approx\frac{h_2+h_4-2h_0}{w^2},$$

where the subscripts of h refer to the numbered nodes of figure 33. Substituting these equivalent expressions for the first two differentials



NOTES:

Filled circles indicate boundary nodes selected to approximate irregular boundary of the area

The grid spacing w is an approximation of the differential distances dx and dy

The five numbered nodes constitute an array

FIGURE 33.—Finite grid and nomenclature used in numerical analysis.

in equation 69, and letting $\partial h/\partial t$ be considered equivalent to $\Delta h_{0t}/\Delta t$, there follows

$$h_1+h_2+h_3+h_4-4h_0=\left[\frac{S}{T}\left(\frac{\Delta h_{0t}}{\Delta_t}\right)-\frac{W}{T}\right]w^2, \qquad (70)$$

where Δh_{0t} is the change in head which occurs at node 0 through the time interval Δt. Depending on the type of data available, equation 70 can be used to compute the ratios S/T and W/T, or the head distribution in time and space, in an aquifer of given size and shape.

Furthermore, the calculations involve only the use of simple arithmetic.

In working out a solution where a finite difference equation such as equation 70 is being applied to a problem in which the head distribution is required, the primary aim of the computing methods is to find the particular head values at all the nodal points (like points 0–4 of figure 33) such that the finite difference equation is satisfied at all nodes simultaneously. This head distribution can be found either by "relaxation" or "iteration" methods of numerical analysis. The computations are begun by assuming head values for all the nodes in the flow system. These initial estimates will not ordinarily satisfy the finite difference equation and will require revision so that the equation will be satisfied at all points in the grid. Refinement of the head values is continued until accuracy is considered to be sufficient for the needs of the problem.

If the refinement is made using a routine adjustment procedure, it is termed an iteration method; if it is made by approximation, it is termed a relaxation method. Consider the steady-state form of equation 70, which is

$$h_1+h_2+h_3+h_4-4h_0+\left(\frac{W}{T}\right)w^2=0. \tag{71}$$

Assume that it is desired to find the head distribution in an aquifer of known size and shape, the flow conditions (and/or head values) being known along its perimeter, and the values of W and T, or their ratio, being known. Substituting in equation 71 the initial set of assumed head values for nodes 0–4, and the known value of $(W/T)w^2$ there follows

$$h_{1i}+h_{2i}+h_{3i}+h_{4i}-4h_{0i}+\left(\frac{W}{T}\right)w^2=R_0, \tag{72}$$

where the subscript i is used to set apart or distinguish the head values that are initially assumed. The term R_0 is the residual at node zero; in other words, it is the remainder resulting from summation of the assumed and known values on the left side of the equation. Inasmuch as it is virtually impossible to assume an initial set of head values such that the summation indicated in equation 71 is zero, there will almost inevitably be a remainder or residual as shown in equation 72. This residual, R_0, may be thought of as an indication of the amount and direction of excess curvature on the piezometric surface defined by the assumed heads. The value of R_0 is also an indication of the amount by which h_{0i} must be changed so that in the next trial summation of head values the result will more closely approach the zero value required for complete satisfaction of equation 71.

It can easily be shown that if the head at the zero node is changed by an amount $\Delta h_0 = -R_0/4$, the residual at that node will be reduced to zero. However, it should be kept in mind that equation 71 is to be tried on each array in the grid net and that residuals will appear at many if not all of the other nodes. Thus a head adjustment at one node will affect residuals at other nodes, and conversely subsequent head adjustments at adjacent nodes will affect the residual at the first zero node. Accordingly, several circuits must be made through the net before the residual values are reduced to zero or nearly zero. Successive circuits of nodal head adjustments amounting to $-R_0/4$, applied regularly over the net, constitute an iterative process. With practice the computer will recognize that the distribution of residuals can be improved more efficiently by either a larger or smaller head adjustment than indicated by $-R_0/4$. Applying such improvement tempered by judgment gained from experience is a relaxation method.

FLOW-NET ANALYSIS

By R. R. Bennett

In analyzing problems of ground-water flow, a graphical representation of the flow pattern is of considerable assistance and often provides the only means of solving those problems for which a mathematical solution is not practicable. The first significant development in graphical analysis of flow patterns was made by Forchheimer (1930).

A "flow net," which is a graphical solution of a flow pattern, is composed of two families of lines or curves. One family of curves represents the streamlines or flow lines, where each curve indicates the path followed by a particle of water as it moves through the aquifer in the direction of decreasing head. Intersecting the streamlines at right angles is a family of curves termed equipotential lines, which represent contours of equal head in the aquifer.

Although the real flow pattern contains an infinite number of flow lines and equipotential lines, it may be conveniently represented by constructing a net that uses only a few of those lines. The flow lines are selected so that the total quantity of flow is divided equally between adjacent pairs of flow lines; similarly the equipotential lines are selected so that the total drop in head across the system is evenly divided between adjacent pairs of potential lines.

The change in potential, or drop in head between two equipotential lines in an aquifer, divided by the distance traversed by a particle of water moving from the higher to the lower potential determines the hydraulic gradient. Recognizing that this movement of a water particle is governed in part by the proposition that the flow path adopted will be the one involving the least work—that is, the shortest possible path between the two equipotential lines in question—it follows that the direction of water movement is everywhere synonymous with paths that are normal to the equipotential lines. Hence the system of flow lines must be drawn orthogonal to the system of equipotential lines.

A flow net constructed with the foregoing principles in mind is a pattern of "rectangles" in which the ratio of the mean dimensions of each "rectangle" is constant. If the net is constructed so that the sides of each rectangle are equal, then the net is a system of "squares." It should be recognized, however, that in flow fields involving curved paths of flow, the elemental geometric forms in the net are curvilinear and thus are not true squares; however, the corners of each "square" are right angles and the mean distances between the two pairs of opposite sides are equal. If any one of these elemental curvilinear "squares" is repeatedly subdivided into four equal parts the subdivisions will progressively approach the shape of true squares.

The proper sketching of a flow net by the graphical method is something of an art that is learned by experience; however, the following points summarized from a paper by Casagrande (1937, p. 136–137) may be helpful to the beginner:

1. Study the appearance of well-constructed flow nets and try to duplicate them by independently reanalyzing the problems they represent.
2. In the first attempts at sketching use only four or five flow channels.
3. Observe the appearance of the entire flow net; do not try to adjust details until the entire net is approximately correct.
4. Frequently parts of a flow net consist of straight and parallel lines, which result in uniformly sized true squares. By starting the sketching in such areas, the solution can be obtained more readily.
5. In flow systems having symmetry (for example, nets depicting radial flow into a well) only a section of the net need be constructed, as the other part or parts are images of that section.
6. During the sketching of the net, keep in mind that the size of the square changes gradually; all transitions are smooth and, where the paths are curved, are of elliptical or parabolic shape.

Taylor (1948) recommends a somewhat different procedure for sketching flow nets. This procedure, called the procedure of explicit trials, has been found to have value in developing intuition for flow-net characteristics. In this method a trial equipotential or flow line is established and the entire net completed as if that trial line were correct. If the completed net is not correct, the initial trial line is resketched and a new net is constructed. The adjustment of the trial line is judged from the appearance of the entire net and how well it conforms to the boundary conditions of the system.

For steady flow, with a particular set of boundary conditions, only one flow net exists. If at some subsequent time the boundary conditions are altered, then after sufficient time has elapsed to reestablish the steady state, a different flow pattern would be developed and again there would be only one possible solution for the new set of boundary conditions. Thus before attempting to construct a flow net, it is important that the boundary conditions be established and carefully described. For example, consider the aquifer shown in figure 34, bounded by an impermeable barrier paralleling a perennial stream. Line AB, designating the stream, is obviously an equipotential line along which the head is equal; line CD marking the impermeable barrier, is evidently coincident with the limiting or boundary flow line. Accordingly, the equipotential lines will adjoin the barrier at right angles. The discharge through any channel or path of the flow net may be obtained with the aid of Darcy's law, one variation of which, as has been previously shown, may be written in the form

$$Q=PIA.$$

For convenience in applying this variation of Darcy's law, consider a unit width or thickness of the aquifer, measured normal to the direction of flow indicated by L in figure 34—that is, normal to the plane of the diagram. The preceding equation may then be rewritten for this unit thickness (in this example) of aquifer, and for one flow channel through the net as

$$\Delta q=PIb, \tag{73}$$

where Δq represents the flow occurring between a pair of adjacent flow lines (one flow channel) and b is the spacing of the flow lines. If L represents the spacing, and Δh the drop in head, between the equipotential lines, equation 73 becomes

$$\Delta q=P\Delta h\left(\frac{b}{L}\right). \tag{74}$$

Inasmuch as the flow net (figure 34) was constructed to comprise a system of "squares" the ratio b/L is equal to unity and the same potential drop occurs across each "square". It follows then from equation 74 that the same incremental flow, Δq, occurs between each pair of adjacent flow lines. If there are n_f flow channels, the total flow, q, through a unit thickness of the aquifer, is given by

$$q=n_f\Delta q. \tag{75}$$

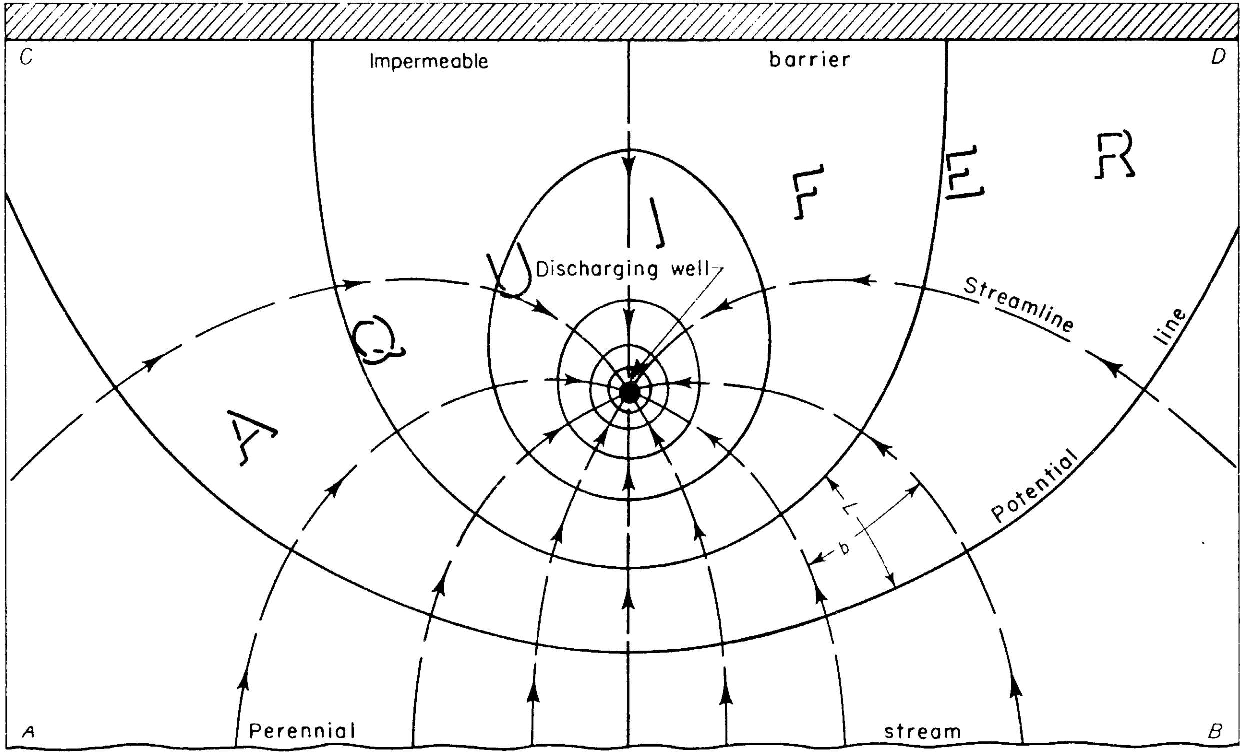


Figure 34.—Flow net for a discharging well in an aquifer bounded by a perennial stream parallel to an impermeable barrier.

If there are n_d potential drops, the total drop in head, h, is given by

$$h = n_d \Delta h. \tag{76}$$

Substituting in equation 75 the values of Δq and Δh given by equations 74 and 76 respectively, there follows

$$q = \frac{n_f}{n_d} Ph. \tag{77}$$

Noting that q represents the total flow through a unit thickness of the aquifer, the equation for total flow through the full thickness of the aquifer becomes

$$Q = \frac{n_f}{n_d} Phm, \tag{78}$$

where Q=flow through full thickness of the aquifer in gallons per day;
n_f=number of flow channels;
n_d=number of potential drops;
P=coefficient of permeability of the aquifer material, in gallons per day per square foot;
m=saturated thickness of aquifer, in feet;
h=total potential drop, in feet; and
Pm=transmissibility of the aquifer, in gallons per day per foot.

The preceding discussion of the graphical construction of flow nets concerns two-dimensional flow fields in homogeneous and isotropic media. The graphical construction of flow nets for three-dimensional problems generally is impracticable; however, many ground-water problems of a three-dimensional nature can be reduced to two dimensions without introducing serious errors.

For two-dimensional problems involving simple anisotropy, such as a constant difference between the vertical and horizontal permeability, or a directional areal transmissibility, the flow net can be constructed by the conventional graphical procedure (system of squares) provided the flow field is first transformed to account for the anisotropy. If the values of maximum or minimum permeability (or transmissibility) are designated P_{max}, and P_{min}, all the dimensions in the direction of P_{max} must be reduced by the factor $\sqrt{P_{min}/P_{max}}$; or, all dimensions in the direction of P_{min} must be increased by the factor $\sqrt{P_{max}/P_{min}}$. After the flow field is transformed, the net is constructed by graphical methods. Then the net is projected back to the original dimensions of the field. It will be found that the projected net generally will no longer be a system of squares, and the equipotential and stream lines will not intersect at right angles.

For areally nonhomogeneous aquifers—that is, those comprising subareas of homogeneous and isotropic media but of different transmissibility—the flow pattern cannot, according to theory, be represented by a single system of squares. If the flow net were constructed so that each flow path conducted the same quantity of water, one subarea could be represented by a system of squares, but the nets in the other subareas would consist of rectangles in which the ratio of the lengths of the sides would be proportional to the differences in transmissibility. If the flow lines from one subarea enter another subarea at an angle, the flow lines (and equipotential lines) would be refracted according to the tangent law. The graphical construction of a flow net under such conditions is extremely difficult and, with the data that are available for most ground-water problems, is generally impossible. However, Bennett and Meyer (1952, p. 54–58) have shown that by generalizing the flow net for such an area into a system of squares and determining the quantity of flow by making an inventory of pumpage in each of the subareas, the approximate transmissibility of the subareas may be determined. Although such an application of the method departs somewhat from theory, it is likely that for many areas it provides more realistic areal transmissibilities than could be obtained by use of pumping-test methods alone. Whereas pumping tests may provide accurate values of transmissibilities they generally represent only a small "sample" of the aquifer. Flow-net analysis on the other hand may include large parts of the aquifer, and hence provide an integrated and more realistic value of the areal transmissibility. Moreover, by including comparatively large parts of the aquifer, the local irregularities that may appreciably affect some pumping-test analyses generally have an insignificant effect on the overall flow patterns.

The application of flow-net analysis to ground-water problems has not received the attention it deserves; however as the versatility of flow-net analysis becomes more widely known, its use will become more common. Such a method of analysis greatly strengthens the hydrologist's insight into ground-water flow systems; it provides quantitative procedures for analyzing and interpreting contour maps of the water-table and piezometric surfaces.

For other illustrations of flow-net construction, see figures 36 and 38.

THEORY OF IMAGES AND HYDROLOGIC BOUNDARY ANALYSIS

The development of the equilibrium and nonequilibrium formulas discussed in the preceding sections was predicated in part on the as-