Theory of Aquifer Tests

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disparity might be due, in part, to the range of stress involved. He states—

> * * * during the pumping test of 1940 the head in the Lloyd sand declined to a new low over a considerable area in the vicinity of the pumped wells, and consequently the stress in the skeleton of the aquifer reached a new high. It is to be expected that the modulus of elasticity would be smaller for the new, higher range of stress than for the old range over which the stress had fluctuated many times.

AQUIFER TESTS—BASIC THEORY

WELL METHODS—POINT SINK OR POINT SOURCE

CONSTANT DISCHARGE OR RECHARGE WITHOUT VERTICAL LEAKAGE

EQUILIBRIUM FORMULA

Wenzel (1942, p. 79–82) showed that the equilibrium formulas used by Slichter (1899), Turneaure and Russell (1901), Israelson (1950), and Wyckoff, Botset, and Muskat (1932) are essentially modified forms of a method developed by Thiem (1906), as are the formulas developed by Dupuit (1848) and Forchheimer (1901). Thiem apparently was the first to use the equilibrium formula for determining permeability and it is frequently associated with his name. The formula was developed by Thiem from Darcy's law and provides a means for determining aquifer transmissibility if the rate of discharge of a pumped well and the drawdown in each of two observation wells at different known distances from the pumped well are known. The Thiem formula, in nondimensional form, can be written as

$$T=\frac{Q\log_e(r_2/r_1)}{2\pi(s_1-s_2)}, \tag{1}$$

where the subscript e in the log term indicates the natural logarithm. In the usual Geological Survey units (see p. 73), and using common logarithms, equation 1 becomes

$$T=\frac{527.7Q\log_{10}(r_2/r_1)}{s_1-s_2}, \tag{2}$$

where

T=coefficient of transmissibility, in gallons per day per foot,
Q=rate of discharge of the pumped well, in gallons per minute,
r_1 and r_2=distances from the pumped well to the first and second observation wells, in feet, and
s_1 and s_2=drawdowns in the first and second observation wells, in feet.

The derivation of the formula is based on the following assumptions: (*a*) the aquifer is homogeneous, isotropic, and of infinite areal extent; (*b*) the discharging well penetrates and receives water from the entire thickness of the aquifer; (*c*) the coefficient of transmissibility is constant at all times and at all places; (*d*) pumping has continued at a uniform rate for sufficient time for the hydraulic system to reach a steady-state (i.e., no change in rate of drawdown as a function of time) condition; and (*e*) the flow is laminar. The formula has wide application to ground-water problems despite the restrictive assumptions on which it is based.

The procedure for application of equation 2 is to select some convenient elapsed pumping time, t, after reaching the steady-state condition, and on semilog coordinate paper plot for each observation well the drawdowns, s, versus the distances, r. By plotting the values of s on the arithmetic scale and the values of r on the logarithmic scale, the observed data should lie on a straight line for the equilibrium formula to apply. From this straight line an arbitrary choice of s_1 and s_2 should be made and the corresponding values of r_1 and r_2 recorded. Equation 2 can then be solved for T.

Jacob (1950, p. 368) recognized that the coefficient of storage could also be determined if the hydraulic system had reached a steady-state condition (see assumption d, above), for thereafter the drawdown is expressed very closely by the nondimensional formula

$$s=\frac{Q}{4\pi T}\log_e\frac{2.25Tt}{r^2S} \qquad (3)$$

or, in the usual Survey units and using common logarithms,

$$s=\frac{264Q}{T}\log_{10}\frac{0.3Tt}{r^2S}. \qquad (4)$$

Thus after the coefficient of transmissibility has been determined, the coordinates of any point on the semilogarithmic graph previously described can be used to solve equation 4 for the coefficient of storage.

NONEQUILIBRIUM FORMULA

Theis (1935) derived the nonequilibrium formula from the analogy between the hydrologic conditions in an aquifer and the thermal conditions in an equivalent thermal system. The analogy between the flow of ground water and heat conduction for the steady-state condition has been recognized at least since the work of Slichter (1899), but Theis was the first to introduce the concept of time to the mathematics of ground-water hydraulics. Jacob (1940) verified the derivation of the nonequilibrium formula directly from hydraulic concepts.

The nonequilibrium formula in nondimensional form is

$$s=\frac{Q}{4\pi T}\int_{r^2S/4Tt}^{\infty}\frac{e^{-u}}{u}\,du, \qquad (5)$$

where $u=r^2S/4Tt$, and where the integral expression is known as an exponential integral.

Using the ordinary Survey units equation 5 may be written as

$$s=\frac{114.6Q}{T}\int_{1.87r^2S/Tt}^{\infty}\frac{e^{-u}}{u}\,du, \tag{6}$$

where

$u=1.87r^2S/Tt$,

s=drawdown, in feet, at any point of observation in the vicinity of a well discharging at a constant rate,

Q=discharge of a well, in gallons per minute,

T=transmissibility, in gallons per day per foot,

r=distance, in feet, from the discharging well to the point of observation,

S=coefficient of storage, expressed as a decimal fraction,

t=time in days since pumping started.

The nonequilibrium formula is based on the following assumptions: (a) the aquifer is homogeneous and isotropic; (b) the aquifer has infinite areal extent; (c) the discharge or recharge well penetrates and receives water from the entire thickness of the aquifer; (d) the coefficient of transmissibility is constant at all times and at all places; (e) the well has an infinitesimal (reasonably small) diameter; and (f) water removed from storage is discharged instantaneously with decline in head. Despite the restrictive assumptions on which it is based, the nonequilibrium formula has been applied successfully to many problems of ground-water flow.

The integral expression in equation 6 cannot be integrated directly, but its value is given by the series

$$\int_{1.87r^2S/Tt}^{\infty}\frac{e^{-u}}{u}\,du=W(u)=-0.577216-\log_e u+u-\frac{u^2}{2\cdot 2!}+\frac{u^3}{3\cdot 3!}-\frac{u^4}{4\cdot 4!}\cdots\cdots \tag{7}$$

where, as already indicated,

$$u=\frac{1.87r^2S}{Tt} \tag{8}$$

The exponential integral is written symbolically as $W(u)$ which is read "well function of u." Values of $W(u)$ for values of u from 10^{-15} to 9.9, as tabulated in Wenzel (1942), are given in table 2. In order to determine the value of $W(u)$ for a given value of u, using table 2, it is necessary to express u as some number (N) between 1.0 and 9.9, multiplied by 10 with the appropriate exponent. For example, when u has a value of 0.0005 (that is, 5.0×10^{-4}), $W(u)$ is determined from the line $N=5.0$ and the column $N\times10^{-4}$ to be 7.0242.

Referring to equations 6 and 8, if s can be measured for one value of r and several values of t, or for one value of t and several values of r, and if the discharge Q is known, then S and T can be determined. Once these aquifer constants have been determined, it is possible, theoretically, to compute the drawdown for any time at any point on the cone of depression for any given rate and distribution of pumping from wells. It is not possible, however, to determine T and S directly from equation 6, because T occurs in the argument of the function and again as a divisor of the exponential integral. Theis devised a convenient graphical method of superposition that makes it possible to obtain a simple solution of the equation.

The first step in this method is the plotting of a type curve on logarithmic coordinate paper. From table 2 values of $W(u)$ have been plotted against the argument u to form the type curve shown in figure 23. It is shown in two segments, A–A and B–B, in order that the portion of the type curve necessary in the analysis of pumping test data could be plotted on a sheet of convenient size. Curve B–B is an extension of curve A–A and overlaps curve A–A for values of $W(u)$ from about 0.22 to 1.0.

Rearranging equations 6 and 8 there follows

$$s=\left[\frac{114.6Q}{T}\right]W(u) \tag{9}$$

or

$$\log s=\left[\log\frac{114.6Q}{T}\right]+\log W(u) \tag{9a}$$

and

$$\frac{r^2}{t}=\left[\frac{T}{1.87S}\right]u \tag{10}$$

or

$$\log\frac{r^2}{t}=\left[\log\frac{T}{1.87S}\right]+\log u \tag{10a}$$

If the discharge, Q, is held constant, the bracketed parts of equations 9a and 10a are constant for a given pumping test, and $W(u)$ is related to

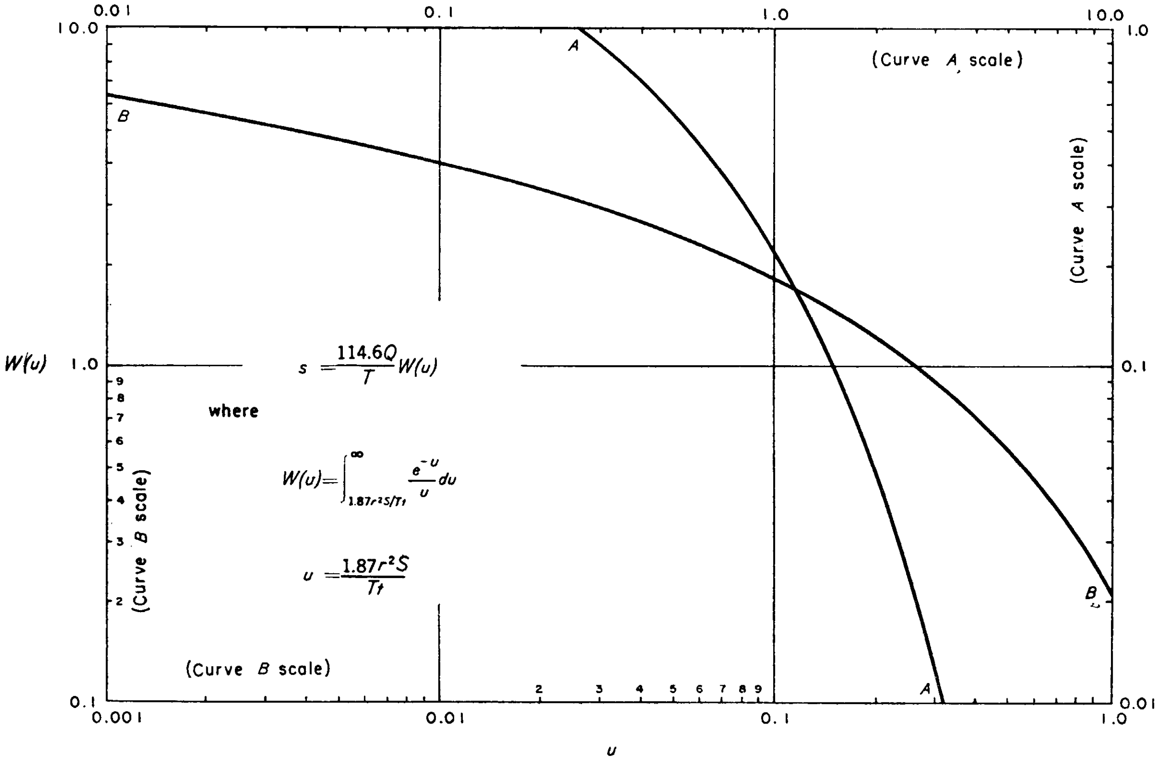


FIGURE 23.—Logarithmic graph of the well function $W(u)$—constant discharge

TABLE 2.—*Values of* W(u) *for values of* u *between* 10^{-15} *and 9.9*

u / N	$N\times10^{-15}$	$N\times10^{-14}$	$N\times10^{-13}$	$N\times10^{-12}$	$N\times10^{-11}$	$N\times10^{-10}$	$N\times10^{-9}$	$N\times10^{-8}$	$N\times10^{-7}$	$N\times10^{-6}$	$N\times10^{-5}$	$N\times10^{-4}$	$N\times10^{-3}$	$N\times10^{-2}$	$N\times10^{-1}$	N
1.0	33.9616	31.6590	29.3564	27.0538	24.7512	22.4486	20.1460	17.8435	15.5409	13.2383	10.9357	8.6332	6.3315	4.0379	1.8229	0.2194
1.1	33.8662	31.5637	29.2611	26.9585	24.6559	22.3533	20.0507	17.7482	15.4456	13.1430	10.8404	8.5379	6.2363	3.9436	1.7371	.1860
1.2	33.7792	31.4767	29.1741	26.8715	24.5689	22.2663	19.9637	17.6611	15.3586	13.0560	10.7534	8.4509	6.1494	3.8576	1.6595	.1584
1.3	33.6992	31.3966	29.0940	26.7914	24.4889	22.1863	19.8837	17.5811	15.2785	12.9759	10.6734	8.3709	6.0695	3.7785	1.5889	.1355
1.4	33.6251	31.3225	29.0199	26.7173	24.4147	22.1122	19.8096	17.5070	15.2044	12.9018	10.5993	8.2968	5.9955	3.7054	1.5241	.1162
1.5	33.5561	31.2535	28.9509	26.6483	24.3458	22.0432	19.7406	17.4380	15.1354	12.8328	10.5303	8.2278	5.9266	3.6374	1.4645	.1000
1.6	33.4916	31.1890	28.8864	26.5838	24.2812	21.9786	19.6760	17.3735	15.0709	12.7683	10.4657	8.1634	5.8621	3.5739	1.4092	.08631
1.7	33.4309	31.1283	28.8258	26.5232	24.2206	21.9180	19.6154	17.3128	15.0103	12.7077	10.4051	8.1027	5.8016	3.5143	1.3578	.07465
1.8	33.3738	31.0712	28.7686	26.4660	24.1634	21.8608	19.5583	17.2557	14.9531	12.6505	10.3479	8.0455	5.7446	3.4581	1.3089	.06471
1.9	33.3197	31.0171	28.7145	26.4119	24.1094	21.8068	19.5042	17.2016	14.8990	12.5964	10.2939	7.9915	5.6906	3.4050	1.2649	.05620
2.0	33.2684	30.9658	28.6632	26.3607	24.0581	21.7555	19.4529	17.1503	14.8477	12.5451	10.2426	7.9402	5.6394	3.3547	1.2227	.04890
2.1	33.2196	30.9170	28.6145	26.3119	24.0093	21.7067	19.4041	17.1015	14.7989	12.4964	10.1938	7.8914	5.5907	3.3069	1.1829	.04261
2.2	33.1731	30.8705	28.5679	26.2653	23.9628	21.6602	19.3576	17.0550	14.7524	12.4498	10.1473	7.8449	5.5443	3.2614	1.1454	.03719
2.3	33.1286	30.8261	28.5235	26.2209	23.9183	21.6157	19.3131	17.0106	14.7080	12.4054	10.1028	7.8004	5.4999	3.2179	1.1099	.03250
2.4	33.0861	30.7835	28.4809	26.1783	23.8758	21.5732	19.2706	16.9680	14.6654	12.3628	10.0603	7.7579	5.4575	3.1763	1.0762	.02844
2.5	33.0453	30.7427	28.4401	26.1375	23.8349	21.5323	19.2298	16.9272	14.6246	12.3220	10.0194	7.7172	5.4167	3.1365	1.0443	.02491
2.6	33.0060	30.7035	28.4009	26.0983	23.7957	21.4931	19.1905	16.8880	14.5854	12.2828	9.9802	7.6779	5.3776	3.0983	1.0139	.02185
2.7	32.9683	30.6657	28.3631	26.0606	23.7580	21.4554	19.1528	16.8502	14.5476	12.2450	9.9425	7.6401	5.3400	3.0615	.9849	.01918
2.8	32.9319	30.6294	28.3268	26.0242	23.7216	21.4190	19.1164	16.8138	14.5113	12.2087	9.9061	7.6038	5.3037	3.0261	.9573	.01686
2.9	32.8968	30.5943	28.2917	25.9891	23.6865	21.3839	19.0813	16.7788	14.4762	12.1736	9.8710	7.5687	5.2687	2.9920	.9309	.01482
3.0	32.8629	30.5604	28.2578	25.9552	23.6526	21.3500	19.0474	16.7449	14.4423	12.1397	9.8371	7.5348	5.2349	2.9591	.9057	.01305
3.1	32.8302	30.5276	28.2250	25.9224	23.6198	21.3172	19.0146	16.7121	14.4095	12.1069	9.8043	7.5020	5.2022	2.9273	.8815	.01149
3.2	32.7984	30.4958	28.1932	25.8907	23.5880	21.2855	18.9829	16.6803	14.3777	12.0751	9.7726	7.4703	5.1706	2.8965	.8583	.01013
3.3	32.7676	30.4651	28.1625	25.8599	23.5573	21.2547	18.9521	16.6495	14.3470	12.0444	9.7418	7.4395	5.1399	2.8668	.8361	.008939
3.4	32.7378	30.4352	28.1326	25.8300	23.5274	21.2249	18.9223	16.6197	14.3171	12.0145	9.7120	7.4097	5.1102	2.8379	.8147	.007891
3.5	32.7088	30.4062	28.1036	25.8010	23.4985	21.1959	18.8933	16.5907	14.2881	11.9855	9.6830	7.3807	5.0813	2.8099	.7942	.006970
3.6	32.6806	30.3780	28.0755	25.7729	23.4703	21.1677	18.8651	16.5625	14.2599	11.9574	9.6548	7.3526	5.0532	2.7827	.7745	.006160
3.7	32.6532	30.3506	28.0481	25.7455	23.4429	21.1403	18.8377	16.5351	14.2325	11.9300	9.6274	7.3252	5.0259	2.7563	.7554	.005448
3.8	32.6266	30.3240	28.0214	25.7188	23.4162	21.1136	18.8110	16.5085	14.2059	11.9033	9.6007	7.2985	4.9993	2.7306	.7371	.004820
3.9	32.6006	30.2980	27.9954	25.6928	23.3902	21.0877	18.7851	16.4825	14.1799	11.8773	9.5748	7.2725	4.9735	2.7056	.7194	.004267
4.0	32.5753	30.2727	27.9701	25.6675	23.3649	21.0623	18.7598	16.4572	14.1546	11.8520	9.5495	7.2472	4.9482	2.6813	.7024	.003779
4.1	32.5506	30.2480	27.9454	25.6428	23.3402	21.0376	18.7351	16.4325	14.1299	11.8273	9.5248	7.2225	4.9236	2.6576	.6859	.003349
4.2	32.5265	30.2239	27.9213	25.6187	23.3161	21.0136	18.7110	16.4084	14.1058	11.8032	9.5007	7.1985	4.8997	2.6344	.6700	.002969
4.3	32.5029	30.2004	27.8978	25.5952	23.2926	20.9900	18.6874	16.3884	14.0823	11.7797	9.4771	7.1749	4.8762	2.6119	.6546	.002633
4.4	32.4800	30.1774	27.8748	25.5722	23.2696	20.9670	18.6644	16.3619	14.0593	11.7567	9.4541	7.1520	4.8533	2.5899	.6397	.002336
4.5	32.4575	30.1549	27.8523	25.5497	23.2471	20.9446	18.6420	16.3394	14.0368	11.7342	9.4317	7.1295	4.8310	2.5684	.6253	.002073
4.6	32.4355	30.1329	27.8303	25.5277	23.2252	20.9226	18.6200	16.3174	14.0148	11.7122	9.4097	7.1075	4.8091	2.5474	.6114	.001841
4.7	32.4140	30.1114	27.8088	25.5062	23.2037	20.9011	18.5985	16.2959	13.9933	11.6907	9.3882	7.0860	4.7877	2.5268	.5979	.001635
4.8	32.3929	30.0904	27.7878	25.4852	23.1826	20.8800	18.5774	16.2748	13.9723	11.6697	9.3671	7.0650	4.7667	2.5068	.5848	.001453
4.9	32.3723	30.0697	27.7672	25.4646	23.1620	20.8594	18.5568	16.2542	13.9516	11.6491	9.3465	7.0444	4.7462	2.4871	.5721	.001291

5.0	32.3521	30.0495	27.7470	25.4444	23.1418	20.8392	18.5366	16.2340	13.9314	11.6289	9.3263	7.0242	4.7261	2.4679	.5598	.001148
5.1	32.3323	30.0297	27.7271	25.4246	23.1220	20.8194	18.5168	16.2142	13.9116	11.6091	9.3065	7.0044	4.7064	2.4491	.5478	.001021
5.2	32.3129	30.0103	27.7077	25.4051	23.1026	20.8000	18.4974	16.1948	13.8922	11.5896	9.2871	6.9850	4.6871	2.4306	.5362	.0009086
5.3	32.2939	29.9913	27.6887	25.3861	23.0835	20.7809	18.4783	16.1758	13.8732	11.5706	9.2681	6.9659	4.6681	2.4126	.5250	.0008086
5.4	32.2752	29.9726	27.6700	25.3674	23.0648	20.7622	18.4596	16.1571	13.8545	11.5519	9.2494	6.9473	4.6495	2.3948	.5140	.0007198
5.5	32.2568	29.9542	27.6516	25.3491	23.0465	20.7439	18.4413	16.1387	13.8361	11.5336	9.2310	6.9289	4.6313	2.3775	.5034	.0006409
5.6	32.2388	29.9362	27.6336	25.3310	23.0285	20.7259	18.4233	16.1207	13.8181	11.5155	9.2130	6.9109	4.6134	2.3604	.4930	.0005708
5.7	32.2211	29.9185	27.6159	25.3133	23.0108	20.7082	18.4056	16.1030	13.8004	11.4978	9.1953	6.8932	4.5958	2.3437	.4830	.0005085
5.8	32.2037	29.9011	27.5985	25.2959	22.9934	20.6908	18.3882	16.0856	13.7830	11.4804	9.1779	6.8758	4.5785	2.3273	.4732	.0004532
5.9	32.1866	29.8840	27.5814	25.2789	22.9763	20.6737	18.3711	16.0685	13.7659	11.4633	9.1608	6.8588	4.5615	2.3111	.4637	.0004039
6.0	32.1698	29.8672	27.5646	25.2620	22.9595	20.6569	18.3543	16.0517	13.7491	11.4465	9.1440	6.8420	4.5448	2.2953	.4544	.0003601
6.1	32.1533	29.8507	27.5481	25.2455	22.9429	20.6403	18.3378	16.0352	13.7326	11.4300	9.1275	6.8254	4.5283	2.2797	.4454	.0003211
6.2	32.1370	29.8344	27.5318	25.2293	22.9267	20.6241	18.3215	16.0189	13.7163	11.4138	9.1112	6.8092	4.5122	2.2645	.4366	.0002864
6.3	32.1210	29.8184	27.5158	25.2133	22.9107	20.6081	18.3055	16.0029	13.7003	11.3978	9.0952	6.7932	4.4963	2.2494	.4280	.0002555
6.4	32.1053	29.8027	27.5001	25.1975	22.8949	20.5923	18.2898	15.9872	13.6846	11.3820	9.0795	6.7775	4.4806	2.2346	.4197	.0002279
6.5	32.0898	29.7872	27.4846	25.1820	22.8794	20.5768	18.2742	15.9717	13.6691	11.3665	9.0640	6.7620	4.4652	2.2201	.4115	.0002034
6.6	32.0745	29.7719	27.4693	25.1667	22.8641	20.5616	18.2590	15.9564	13.6538	11.3512	9.0487	6.7467	4.4501	2.2058	.4036	.0001816
6.7	32.0595	29.7569	27.4543	25.1517	22.8491	20.5465	18.2439	15.9414	13.6388	11.3362	9.0337	6.7317	4.4351	2.1917	.3959	.0001621
6.8	32.0446	29.7421	27.4395	25.1369	22.8343	20.5317	18.2291	15.9265	13.6240	11.3214	9.0189	6.7169	4.4204	2.1779	.3883	.0001448
6.9	32.0300	29.7275	27.4249	25.1223	22.8197	20.5171	18.2145	15.9119	13.6094	11.3608	9.0043	6.7023	4.4059	2.1643	.3810	.0001293
7.0	32.0156	29.7131	27.4105	25.1079	22.8053	20.5027	18.2001	15.8976	13.5950	11.2924	8.9899	6.6879	4.3916	2.1508	.3738	.0001155
7.1	32.0015	29.6989	27.3963	25.0937	22.7911	20.4885	18.1860	15.8834	13.5808	11.2782	8.9757	6.6737	4.3775	2.1376	.3668	.0001032
7.2	31.9875	29.6849	27.3823	25.0797	22.7771	20.4746	18.1720	15.8694	13.5668	11.2642	8.9617	6.6598	4.3636	2.1246	.3599	.00009219
7.3	31.9737	29.6711	27.3685	25.0659	22.7633	20.4608	18.1582	15.8556	13.5530	11.2504	8.9479	6.6460	4.3500	2.1118	.3532	.00008239
7.4	31.9601	29.6575	27.3549	25.0523	22.7497	20.4472	18.1446	15.8420	13.5394	11.2368	8.9343	6.6324	4.3364	2.0991	.3467	.00007364
7.5	31.9467	29.6441	27.3415	25.0389	22.7363	20.4337	18.1311	15.8286	13.5260	11.2234	8.9209	6.6190	4.3231	2.0867	.3403	.00006583
7.6	31.9334	29.6308	27.3282	25.0257	22.7231	20.4205	18.1179	15.8153	13.5127	11.2102	8.9076	6.6057	4.3100	2.0744	.3341	.00005886
7.7	31.9203	29.6178	27.3152	25.0126	22.7100	20.4074	18.1048	15.8022	13.4997	11.1971	8.8946	6.5927	4.2970	2.0623	.3280	.00005263
7.8	31.9074	29.6048	27.3023	24.9997	22.6971	20.3945	18.0919	15.7893	13.4868	11.1842	8.8817	6.5798	4.2842	2.0503	.3221	.00004707
7.9	31.8947	29.5921	27.2895	24.9869	22.6844	20.3818	18.0792	15.7766	13.4740	11.1714	8.8689	6.5671	4.2716	2.0386	.3163	.00004210
8.0	31.8821	29.5795	27.2769	24.9744	22.6718	20.3692	18.0666	15.7640	13.4614	11.1589	8.8563	6.5545	4.2591	2.0269	.3106	.00003767
8.1	31.8697	29.5671	27.2645	24.9619	22.6594	20.3568	18.0542	15.7516	13.4490	11.1464	8.8439	6.5421	4.2468	2.0155	.3050	.00003370
8.2	31.8574	29.5548	27.2523	24.9497	22.6471	20.3445	18.0419	15.7393	13.4367	11.1342	8.8317	6.5298	4.2346	2.0042	.2996	.00003015
8.3	31.8453	29.5427	27.2401	24.9375	22.6350	20.3324	18.0298	15.7272	13.4246	11.1220	8.8195	6.5177	4.2226	1.9930	.2943	.00002699
8.4	31.8333	29.5307	27.2282	24.9256	22.6230	20.3204	18.0178	15.7152	13.4126	11.1101	8.8076	6.5057	4.2107	1.9820	.2891	.00002415
8.5	31.8215	29.5189	27.2163	24.9137	22.6112	20.3086	18.0060	15.7034	13.4008	11.0982	8.7957	6.4939	4.1990	1.9711	.2840	.00002162
8.6	31.8098	29.5072	27.2046	24.9020	22.5995	20.2969	17.9943	15.6917	13.3891	11.0865	8.7840	6.4822	4.1874	1.9604	.2790	.00001936
8.7	31.7982	29.4957	27.1931	24.8905	22.5879	20.2853	17.9827	15.6801	13.3776	11.0750	8.7725	6.4707	4.1759	1.9498	.2742	.00001733
8.8	31.7868	29.4842	27.1816	24.8790	22.5765	20.2739	17.9713	15.6687	13.3661	11.0635	8.7610	6.4592	4.1646	1.9393	.2694	.00001552
8.9	31.7755	29.4729	27.1703	24.8678	22.5652	20.2626	17.9600	15.6574	13.3548	11.0523	8.7497	6.4480	4.1534	1.9290	.2647	.00001390
9.0	31.7643	29.4618	27.1592	24.8566	22.5540	20.2514	17.9488	15.6462	13.3437	11.0411	8.7386	6.4368	4.1423	1.9187	.2602	.00001245
9.1	31.7533	29.4507	27.1481	24.8455	22.5429	20.2404	17.9378	15.6352	13.3326	11.0300	8.7275	6.4258	4.1313	1.9087	.2557	.00001115
9.2	31.7424	29.4398	27.1372	24.8346	22.5320	20.2294	17.9268	15.6243	13.3217	11.0191	8.7166	6.4148	4.1205	1.8987	.2513	.000009988
9.3	31.7315	29.4290	27.1264	24.8238	22.5212	20.2186	17.9160	15.6135	13.3109	11.0083	8.7058	6.4040	4.1098	1.8888	.2470	.000008948
9.4	31.7208	29.4183	27.1157	24.8131	22.5105	20.2079	17.9053	15.6028	13.3002	10.9976	8.6951	6.3934	4.0992	1.8791	.2429	.000008018
9.5	31.7103	29.4077	27.1051	24.8025	22.4999	20.1973	17.8948	15.5922	13.2896	10.9870	8.6845	6.3828	4.0887	1.8695	.2387	.000007185
9.6	31.6998	29.3972	27.0946	24.7920	22.4895	20.1869	17.8843	15.5817	13.2791	10.9765	8.6740	6.3723	4.0784	1.8599	.2347	.000006439
9.7	31.6894	29.3868	27.0843	24.7817	22.4791	20.1765	17.8739	15.5713	13.2688	10.9662	8.6637	6.3620	4.0681	1.8505	.2308	.000005771
9.8	31.6792	29.3766	27.0740	24.7714	22.4688	20.1663	17.8637	15.5611	13.2585	10.9559	8.6534	6.3517	4.0579	1.8412	.2269	.000005173
9.9	31.6690	29.3664	27.0639	24.7613	22.4587	20.1561	17.8535	15.5509	13.2483	10.9458	8.6433	6.3416	4.0479	1.8320	.2231	.000004637

u in the manner that s is related to r^2/t. This is shown graphically in figure 24. Therefore, if values of the drawdown s are plotted against r^2/t, or $1/t$ if only one observation well is used, on logarithmic tracing paper to the same scale as the type curve, the curve of observed data will be similar to the type curve. The data curve may then be superposed on the type curve, the coordinate axes of the two curves being held parallel, and translated to a position which represents the best fit of the field data to the type curve. An arbitrary point is selected anywhere on the overlapping portion of the sheets and the coordinates of this common point on both sheets are recorded. It is often convenient to select a point whose coordinates are both 1. These data are then used with equations 9 and 10 to solve for T and S.

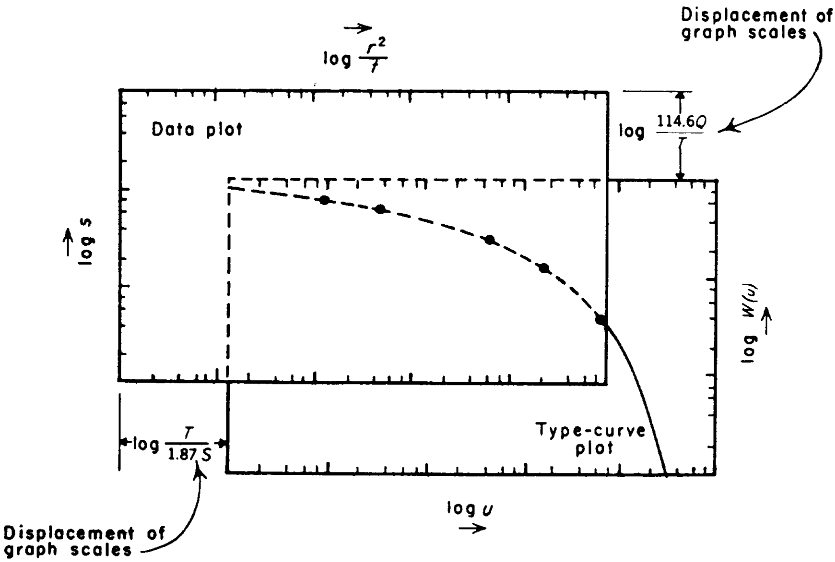


FIGURE 24.—Relation of $W(u)$ and u to s and r^2/t.

A type curve on logarithmic coordinate paper of $W(u)$ versus $1/u$, the reciprocal of the argument, could have been plotted. Values of the drawdown (or recovery), s, would then have been plotted versus t, or t/r^2 and superposed on the type curve in the manner outlined above. This method eliminates the necessity for computing $1/t$ values for the values of s.

MODIFIED NONEQUILIBRIUM FORMULA

It was recognized by Jacob (1950) that in the series of equation 7 the sum of the terms beyond $\log_e u$ is not significant when u becomes small. The value of u decreases as the time, t, increases and as r decreases. Therefore, for large values of t and reasonably small values of r, the terms beyond $\log_e u$ in equation 7 may be neglected. When r is large, t must be very large before the terms beyond $\log_e u$ in equation 7 can be neglected. Thus the Theis equation in its abbreviated or modified nondimensional form is written as

$$s=\frac{Q}{4\pi T}\left(\log_e\frac{4Tt}{r^2S}-0.5772\right)$$

$$=\frac{Q}{4\pi T}\log_e\frac{2.25Tt}{r^2S},$$

which is obviously identical with equation 3. In the usual Survey units, then, this equation will be identical with equation 4, all terms being as previously defined.

In applying equation 4 to measurements of the drawdown or recovery of water level in a particular observation well, the distance r will be constant, and it follows that

$$\text{at time } t_1,\ s_1=\frac{264Q}{T}\left(\log_{10}\frac{0.3Tt_1}{r^2S}\right);$$

$$\text{at time } t_2,\ s_2=\frac{264Q}{T}\left(\log_{10}\frac{0.3Tt_2}{r^2S}\right);$$

and the change in drawdown or recovery from time t_1 to t_2 is

$$s_2-s_1=\frac{264Q}{T}\left(\log_{10}\frac{t_2}{t_1}\right).$$

Rewriting this equation in form suitable for direct solution of T, there follows

$$T=\frac{264Q(\log_{10} t_2/t_1)}{s_2-s_1}, \tag{11}$$

where Q and T are as previously defined, t_1 and t_2 are two selected times, in any convenient units, since pumping started or stopped, and s_1 and s_2 are the respective drawdowns or recoveries at the noted times, in feet.

The most convenient procedure for application of equation 11 is to plot the observed data for each well on the semilogarithmic coordinate paper, plotting values of t on the logarithmic scale and values of s on the arithmetic scale. After the value of u becomes small (generally less than 0.01) and the value of time, t, becomes great, the observed data should fall on a straight line. From this straight line make an arbitrary choice of t_1 and t_2 and record the corresponding values of s_1 and s_2. Equation 11 can then be solved for T. For convenience, t_1 and t_2 are usually chosen one log cycle apart, because then

$$\log_{10}\frac{t_2}{t_1}=1$$

and equation 11 reduces to

$$T=\frac{264Q}{\Delta s}, \tag{12}$$

where Δs is the change, in feet, in the drawdown or recovery over one log cycle of time.

The coefficient of storage also can be determined from the same semilog plot of the observed data. When $s=0$, equation 3 becomes

$$s=0=\frac{Q}{4\pi T}\log_e\frac{2.25Tt}{r^2S}.$$

Solving for the coefficient of storage, S, the equation in its final form becomes

$$S=\frac{2.25Tt}{r^2} \tag{13}$$

or, in the usual Survey units,

$$S=\frac{0.3Tt_0}{r^2}, \tag{14}$$

where S, T, and r are as previously defined and t_0 is the time intercept, in days, where the plotted straight line intersects the zero-drawdown axis. If any other units were used for the time, t, on the semilog plot, then obviously t_0 must be converted to days before using equation 14. Lohman (1957) has described a simple method for determining S using the data region of the straight-line plot without extrapolating to the zero-drawdown axis.

THEIS RECOVERY FORMULA

A useful corollary to the nonequilibrium formula was devised by Theis (1935) for the analysis of the recovery of a pumped well. If a well is pumped, or allowed to flow, for a known period of time and then shut down and allowed to recover, the residual drawdown at any instant will be the same as if the discharge of the well had been continued but a recharge well with the same flow had been introduced at the same point at the instant the discharge stopped. The residual drawdown at any time during the recovery period is the difference between the observed water level and the nonpumping water level extrapolated from the observed trend prior to the pumping period, The residual drawdown, s', at any instant will then be

$$s'=\frac{114.6Q}{T}\left[\int_{1.87r^2S/Tt}^{\infty}\frac{e^{-u}}{u}du-\int_{1.87r^2S/Tt'}^{\infty}\frac{e^{-u}}{u}du\right] \qquad (15)$$

where Q, T, S, and r are as previously defined, t is the time since pumping started, and t' is the time since pumping stopped. The quantity $1.87r^2S/Tt'$ will be small when t' ceases to be small because r is very small and therefore the value of the integral will be given closely by the first two terms of the infinite series of equation 7, Equation 15 can therefore be written, in modified form, in the usual Survey units, as

$$T=\frac{264Q}{s'}\log_{10}\frac{t}{t'} \qquad (16)$$

The above formula is similar in form to, and is based on the same assumptions as, the modified nonequilibrium formula developed by Jacob, and it permits the computation of the coefficient of transmissibility of an aquifer from the observation of the rate of recovery of water level in a pumped well, or in a nearby observation well where r is sufficiently small to meet the above assumptions.

The Theis recovery formula is applied in much the same manner as the modified nonequilibrium formula. The most convenient procedure is to plot the residual drawdown, s', against t/t' on semilogarithmic coordinate paper, s' being plotted on the arithmetic scale and t/t' on the logarithmic scale. After the value of t' becomes sufficiently large, the observed data should fall on a straight line. The slope of this line gives the value of the quantity $\log_{10}(t/t')/s'$ in equation 16. For convenience, the value of t/t' is usually chosen over one log cycle because its logarithm is then unity and equation 16 then reduces to

$$T=\frac{264\,Q}{\Delta s'} \qquad (17)$$

where $\Delta s'$ is the change in residual drawdown, in feet, per log cycle of time. It is not possible to determine the coefficient of storage from the observation of the rate of recovery of a pumped well unless the effective radius, r_w, which is usually difficult to determine, is known. The Theis recovery formula should be used with caution in areas where it is suspected that boundary conditions exist. If a geologic boundary has been intercepted by the cone of depression during pumping, it may be reflected in the rate of recovery of the pumped well, and the value of T determined by using the Theis recovery formula could be in error. With reasonable care the recovery in an observation well can be used, of course, to determine both transmissibility and storage, whether or not boundaries are present.

APPLICABILITY OF METHODS TO ARTESIAN AND WATER-TABLE AQUIFERS

The methods previously discussed have been used successfully for many years in determining aquifer constants and in predicting the performance of both water-table and artesian aquifers. The derivations of the equations are based, in part, on the assumptions that the coefficient of transmissibility is constant at all times and places and that water is released from storage instantaneously with decline in head. It should be recognized, however, that these and many other idealizations are necessary before mathematical models can be used to analyze the physical phenomena associated with ground-water movement. Thus the hydrologist cannot blindly select a model, turn a crank, and accept the answers. He must devote considerable time and thought to judging how closely his real aquifer resembles the ideal. If enough data are available he will always find that no ideal aquifer, of the type postulated in the theory, could reproduce the data obtained in an actual pumping test. He should understand that the dispersion of the data is a measure of how far his aquifer departs from the ideal. Therefore, he must plan his test procedures so that they will conform as closely as possible to the theory and thus give results that can safely be applied to his aquifer. He must be prepared to find out, however, that his aquifer is too complex to permit a clear evaluation of its coefficients of transmissibility and storage. He must not tell himself or the reader that "the coefficient of storage changed" during the test but must realize that he got different values when he tried to apply his data, inconsistently, to an ideal theoretical aquifer.

Thus there is little justification for the premise that the storage coefficient of a water-table aquifer varies with the time of pumping, inasmuch as such anomalous data are merely the results of trying to apply a two-dimensional flow formula to a three-dimensional problem. The nonequilibrium formula was derived on the basis of strictly radial flow in an infinite aquifer and its application to situations where vertical-flow components occur is not justified except under certain limiting conditions. As the time of pumping becomes large, however, the rate of water-level decline decreases rapidly so that eventually the effect of vertical-flow components in water-table aquifers are minimized.

If the drawdowns are large compared to the initial depth of flow, it is necessary to adjust the observed drawdown in a pumping test of a water-table aquifer before the nonequilibrium formula is applied. According to Jacob (1944, p. 4) if the observed drawdowns are adjusted (reduced) by the factor $s^2/2m$, where s is the observed drawdown and m is the initial depth of flow, the value of T will correspond to equivalent confined flow of uniform depth, and the value of S will more closely approximate the true value. He adds that when the drawdowns are adjusted the nonequilibrium formula can be used with fair assurance even when the dewatering is as much as 25 percent of the initial depth of flow.

Where the discharging well only partially penetrates the aquifer it may also be necessary to adjust the observed drawdowns. Procedures for accomplishing this have been described by Jacob (1945).

INSTANTANEOUS DISCHARGE OR RECHARGE

"BAILER" METHOD

Skibitzke (1958) has developed a method for determining the coefficient of transmissibility from the recovery of the water level in a well that has been bailed. At any given point on the recovery curve the following equation applies:

$$s'=\frac{V}{4\pi Tt\left[e^{r_w{}^2S/4Tt}\right]} \tag{18}$$

where

s'=residual drawdown,
V=volume of water removed in one bailer cycle,
T=coefficient of transmissibility,
S=coefficient of storage,
t=length of time since the bailer was removed,
r_w=effective radius of the well.

The effective radius, r_w, of the well is very small in comparison to the extent of the aquifer. As r_w is small, the term in brackets in equation 18 approaches unity as t increases. Therefore for large values of t, equation 18 may be modified and rewritten, in consistent units, as

$$s'=\frac{V}{4\pi Tt}=\frac{V}{12.57Tt}, \tag{19}$$

where s', T, and t have units and significance as previously defined, and where V represents the volume of water, in gallons, removed during one bailer cycle. If the residual drawdown is observed at some time after completion of n bailer cycles then the following expression applies:

$$s'=\frac{1}{12.57T}\left[\frac{V_1}{t_1}+\frac{V_2}{t_2}+\frac{V_3}{t_3}+\cdots\frac{V_n}{t_n}\right], \tag{20}$$

where the subscripts merely identify each cycle of events in sequence.

Thus V_3 represents the volume of water removed during the third bailer cycle and t_3 is the elapsed time from the instant that water was removed from storage to the instant at which the observation of residual drawdown was made.

If approximately the same volume of water is removed by the bailer during each cycle, then equation 20 becomes

$$s'=\frac{V}{12.57T}\left[\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\cdots\frac{1}{t_n}\right]. \tag{21}$$

The "bailer" method is thus applied to a single observation of the residual drawdown after the time since bailing stopped becomes large. The transmissibility is computed by substituting in equation 21 the observed residual drawdown, the volume of water V considered to be the average amount removed by the bailer in each cycle, and the summation of the reciprocal of the elapsed time, in days, between the time each bailer of water was removed from the well and the time of observation of residual drawdown.

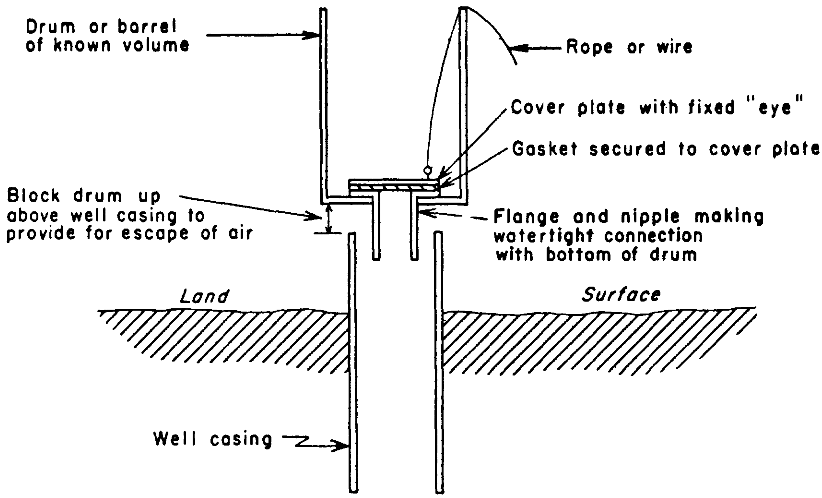
"SLUG" METHOD

Ferris and Knowles (1954) discuss a convenient method for estimating the coefficient of transmissibility, under certain conditions. This is done by injecting a given quantity or "slug" of water into a well. Their equation for determining the coefficient of transmissibility is the same as the equation derived by Skibitzke for the bailer method, inasmuch as the effects of injecting a slug of water into a well are identical, except for sign, with the effects of bailing out a slug of water. Thus equation 19 has direct application, only s' now represents residual head, in feet, at the time t, in days, following injection of V gallons of water.

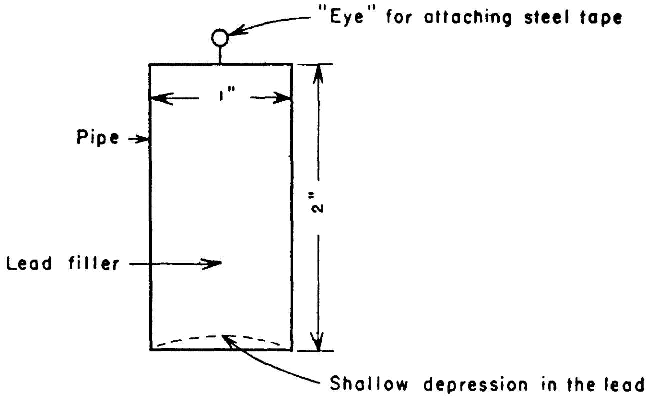
As used in the field, this method requires the sudden injection of a known volume of water into a well and the collection thereafter of a rapid series of water-level observations to define the decay of the head that was built up in the well. An arithmetic plot of residual head values versus the reciprocals of the times of observation should produce a straight line whose slope, appropriately substituted in equation 19, permits computation of the transmissibility.

Suggested equipment for use in injecting a slug of water into a well, and for making the rapid series of water-level observations required immediately thereafter, is shown schematically in figure 25.

The duration of a "slug" test is very short, hence the estimated transmissibility determined from the test will be representative only of the water-bearing material close to the well. Serious errors will



A. APPARATUS FOR MAKING "SLUG" TEST



B. PLAN FOR PERCUSSION INSTRUMENT FOR RAPID MEASUREMENT OF WATER LEVELS

FIGURE 25.—Suggested equipment for a "slug" test.

be introduced unless the observation well is fully developed and completely penetrates the aquifer. Use of the "slug" test should probably be restricted to artesian aquifers of small to moderate transmissibility (less than 50,000 gallons per day per foot).

CONSTANT HEAD WITHOUT VERTICAL LEAKAGE

Controlled pumping tests have proved to be an effective tool in determining the coefficients of storage and transmissibility. In the usual test the discharge rate of the pumped well is held constant, whereas the drawdown varies with time. The resulting data are analyzed graphically as previously described. Jacob and Lohman (1952) derived a formula for determining the coefficients of storage and transmissibility from a test in which the discharge varies with time and the drawdown is held constant. The formula, based on the assumptions that the aquifer is of infinite areal extent, and that the coefficients of transmissibility and storage are constant at all times and all places, is developed from the analogy between the hydrologic conditions in an aquifer and the thermal conditions in an equivalent thermal system. The formula is written as

$$Q=2\pi T s_w G(\alpha), \tag{22}$$

where

$$G(\alpha)=\frac{4\alpha}{\pi}\int_0^\infty xe^{-\alpha x^2}\left[\frac{\pi}{2}+\tan^{-1}\frac{Y_0(x)}{J_0(x)}\right]dx \tag{23}$$

and

$$\alpha=\frac{Tt}{r_w^2 S}. \tag{24}$$

Using the customary Survey units, equations 22 and 24 are rewritten in the form

$$Q=\frac{Ts_wG(\alpha)}{229} \tag{25}$$

and

$$\alpha=\frac{0.134\ Tt}{r_w^2 S} \tag{26}$$

where Q, T, and t have the units and meaning previously defined and where

s_w=constant drawdown, in feet, in the discharging well,

r_w=effective radius, in feet, of the discharging well.

The terms $J_0(x)$ and $Y_0(x)$ are Bessel functions of zero order of the first and second kinds respectively.

The integration required in equation 23 cannot be accomplished directly so it is necessary to replace the integral with a summation and solve it by numerical methods. In this fashion values of $G(\alpha)$ for values of α from 10^{-4} to 10^{12}, have been tabulated by Jacob and Lohman, (1952), and are given herewith in table 3. The term $G(\alpha)$ is here designated as the "well function of α, constant-head situation." This table is used in the same manner as table 2, which gives values of $W(u)$ versus u.

It is seen from equations 25 and 26 that if Q can be measured for several values of t and if the constant drawdown, s_w, and the effective radius, r_w, are known, S and T can be determined. It is not possible to determine S and T directly, however, since T occurs both in the argument of the function and as a multiplier of $G(\alpha)$. A convenient graphical method, similar to that used in solving the nonequilibrium formula, makes it possible to obtain a simple solution.

The first step in this method is the plotting of a type curve on logarithmic coordinate paper. From table 3, values of $G(\alpha)$ were plotted against the argument α to form the type curve shown in figure 26. It is shown in several segments in order that the entire type curve may be plotted on a sheet of convenient size.

Rearranging equations 25 and 26 there follows:

$$Q=\frac{Ts_w}{229}G(\alpha)$$

or

$$\log Q=\left[\log\frac{Ts_w}{229}\right]+\log G(\alpha), \qquad (27)$$

and

$$t=\frac{r_w^2S}{0.13T}\alpha$$

or

$$\log t=\left[\log\frac{r_w^2S}{0.13T}\right]+\log \alpha. \qquad (28)$$

If the drawdown, s_w, is held constant, the bracketed parts of equations 27 and 28 are constant for any given test and log $G(\alpha)$ is related to log α in the same manner that log Q is related to log t. (Note the similarity in form between equations 27 and 28 and equations 9a and 10a.) Therefore if values of the discharge, Q, are plotted against corresponding values of time, t, on logarithmic tracing paper to the same scale as the type curve, the curve of observed data will be similar to the type curve. The data curve may then be superposed on the type curve, the coordinated axes of the two curves being held parallel, and translated to a position that represents the best fit of the data to the type curve. An arbitrary point is selected on the overlapping portion of the sheets and the coordinates of this common point on both sheets are used with equations 25 and 26 to solve for T and S. This graphical solution is similar to that used with the Theis nonequilibrium formula.

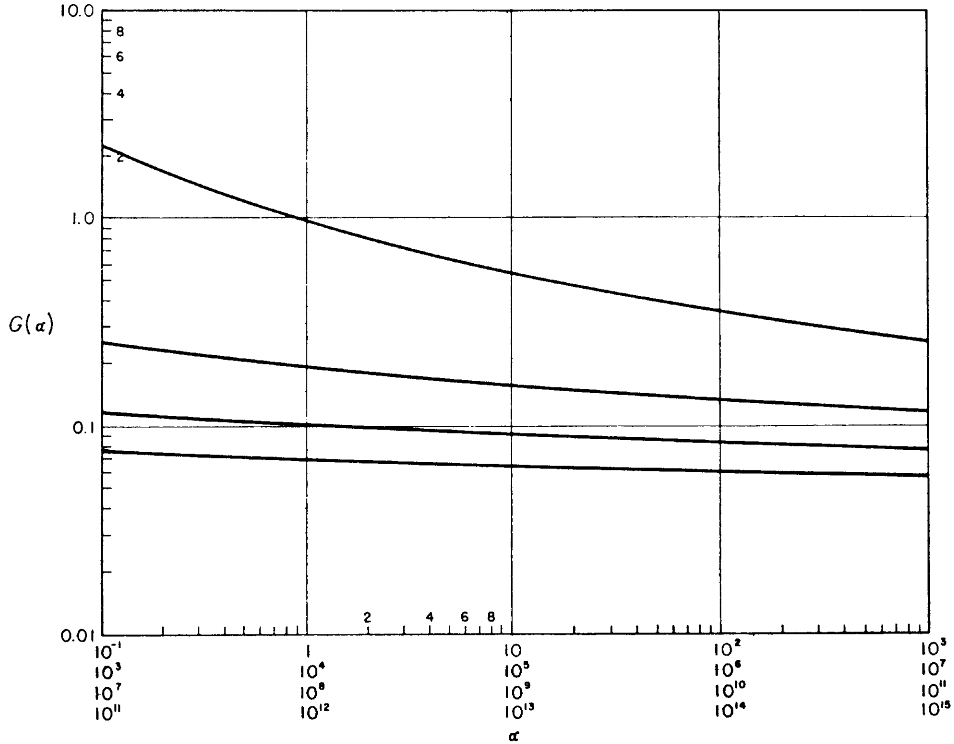


FIGURE 26.—Logarithmic graph of well function $G(\alpha)$—constant drawdown.

TABLE 3.—*Values of* G(α) *for values of* α *between* 10^{-4} *and* 10^{12}

[From Jacob and Lohman, 1952, p. 561]

	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}
1	56.9	18.34	6.13	2.249	0.985	0.534	0.346	0.251
2	40.4	13.11	4.47	1.716	.803	.461	.311	.232
3	33.1	10.79	3.74	1.477	.719	.427	.294	.222
4	28.7	9.41	3.30	1.333	.667	.405	.283	.215
5	25.7	8.47	3.00	1.234	.630	.389	.274	.210
6	23.5	7.77	2.78	1.160	.602	.377	.268	.206
7	21.8	7.23	2.60	1.103	.580	.367	.263	.203
8	20.4	6.79	2.46	1.057	.562	.359	.258	.200
9	19.3	6.43	2.35	1.018	.547	.352	.254	.198
10	18.3	6.13	2.25	.985	.534	.346	.251	.196
	10^{4}	10^{5}	10^{6}	10^{7}	10^{8}	10^{9}	10^{10}	10^{11}
1	0.1964	0.1608	0.1360	0.1177	0.1037	0.0927	0.0838	0.0764
2	.1841	.1524	.1299	.1131	.1002	.0899	.0814	.0744
3	.1777	.1479	.1266	.1106	.0962	.0883	.0801	.0733
4	.1733	.1449	.1244	.1089	.0968	.0872	.0792	.0726
5	.1701	.1426	.1227	.1076	.0958	.0864	.0785	.0720
6	.1675	.1408	.1213	.1066	.0950	.0857	.0779	.0716
7	.1654	.1393	.1202	.1057	.0943	.0851	.0774	.0712
8	.1636	.1380	.1192	.1049	.0937	.0846	.0770	.0709
9	.1621	.1369	.1184	.1043	.0932	.0842	.0767	.0706
10	.1608	.1360	.1177	.1037	.0927	.0838	.0764	.0704

Jacob and Lohman (1952) showed that for large values of t, the function $G(\alpha)$ can be replaced by $2/W(u)$, and it has already been shown (see discussion, p. 99) that the approximate form of $W(u)$ is given by 2.30 $\log_{10}$ $(2.25Tt/Sr_w^2)$. Making this substitution for $G(\alpha)$ in equation 22, there follows

$$Q=\frac{4\pi T s_w/2.30}{\log_{10}(2.25Tt/r_w^2S)}$$

or, rearranging terms,

$$\frac{s_w}{Q}=\frac{2.30}{4\pi T}\log\frac{t}{r_w^2}+\frac{2.30}{4\pi T}\log\frac{2.25T}{S}. \qquad (29)$$

It should be evident from the form of equation 29, that if arithmetic values of the variable s_w/Q are plotted against logarithmic values of the variable t/r_w^2 the points will define a straight line. The slope of this line, in equation 29, is the prefix of the variable term log (t/r_w^2). In other words,

$$\text{Slope of straight-line plot}=\frac{\Delta(s_w/Q)}{\Delta\log(t/r_w^2)}=\frac{2.30}{4\pi T}.$$

Once the slope of the graph is determined, therefore, the coefficient of transmissibility may be computed from the relation

$$T=\frac{2.30\Delta(\log t/r_w^2)}{4\pi\Delta(s_w/Q)}. \qquad (30)$$

If the slope is measured over one log cycle then the term $\Delta \log (t/r_w^2)$ equals unity and equation 30 is further simplified to the form

$$T=\frac{2.30}{4\pi\Delta(s_w/Q)}. \tag{31}$$

The coefficient of storage could then be found by substituting in equation 29 the computed value of T and the coordinates of any convenient point on the straight-line plot. However, the computation is greatly simplified by noting that for the point where the straight-line plot intersects the logarithmic time axis (that is, where $s_w/Q=0$), equation 29 becomes

$$S=2.25T(t/r_w^2)_0. \tag{32}$$

In the usual Survey units, equations 31 and 32 are written

$$T=\frac{264}{\Delta(s_w/Q)} \tag{33}$$

and

$$S=0.3T(t/r_w^2)_0. \tag{34}$$

Thus equations 33 and 34 are applied through the simple device of a semilogarithmic plot where values of s_w/Q are plotted on the arithmetic scale against corresponding values of t/r_w^2 on the logarithmic scale.

The methods that have been outlined in this section are useful in determining the coefficient of transmissibility but should be used with caution in determining the coefficient of storage because it is often difficult to determine the effective radius of the pumped well.

CONSTANT DISCHARGE WITH VERTICAL LEAKAGE

"LEAKY AQUIFER" FORMULA

A problem of practical interest is that of an elastic artesian aquifer that is replenished by vertical leakage through overlying or underlying semipermeable confining beds. In most places the confining beds only impede or retard the movement of ground water rather than prevent it. It is often true that this retardation of ground-water movement is sufficient so that the Theis equation (which assumes impermeable confining beds) can be applied. Nevertheless there will be occasions when departure of the test data from the predictions of the Theis equation will require investigation of the ability of the confining beds to transmit water.

As an example of the magnitude of flow through material of low permeability, consider a semipermeable confining bed, 50 feet thick, consisting of silty clay that has a permeability of 0.2 gallon per day per square foot. Such a material is listed by Wenzel (1942, p. 13, lab. no. 2,278) as including about 49 percent (by weight) clay and about 45 percent silt. Assume that the confining bed is saturated and that in some manner there is established and maintained a head differential of 25 feet between the top and bottom surfaces of the bed. The rate of percolation, related to this head differential, through the confining bed is computed from the previously given (see p. 73) variant of Darcy's law,

$$Q_d = P'IA,$$

where, in this example,

Q_d=discharge in gallons per day through specified area of confining bed,

P'=vertical permeability of confining bed=0.2 gallon per day per square foot,

I=hydraulic gradient imposed on confining bed=25/50=0.5 foot per foot.

A=specified area of confining bed through which percolation occurs.

Thus, through a confining-bed area of one square foot,

$$Q_d = 0.2 \times 0.5 \times 1 = 0.1 \text{ gallon per day,}$$

or, through a confining-bed area of one square mile,

$$Q_d = 0.2 \times 0.5 \times 5{,}280 \times 5{,}280 = 2{,}800{,}000 \text{ gallons per day.}$$

It is known that the cone of depression created by pumping a well in an artesian aquifer grows rapidly and thus in a relatively short time encompasses a large area. As shown by the above computations, the total amount of vertical seepage through confining beds may be quite large, even though the permeability of these formations is relatively small. If the confining bed in turn is overlain by an aquifer of appreciable storage and transmitting capacity, the radius of the cone of influence developed by a well pumping from the artesian aquifer will be determined by the hydrologic regimen of the artesian aquifer, the confining bed, and the leakage-source aquifer.

The first detailed analysis and solution of the leaky-aquifer problem was developed by DeGlee [1] (1930) and later supplemented by Steggewentz and Van Nes (1939).

In these analyses, assumptions related to the physical flow system are: (a) the artesian aquifer is bounded above or below by a semipermeable confining bed, (b) the aquifer, when pumped is supplied by leakage through the confining bed, the leakage being proportional to the drawdown, and (c) the aquifer and confining bed are independently homogeneous and isotropic. It is also assumed that the water level in the aquifer supplying water to the semipermeable bed is maintained

[1] Glee, G. J. de, 1930, over grondwaterstroomingen bei wateronttrekking door mittel van putten [On ground-water currents through draining by means of wells]: Delft [Netherlands] Tech. Hogeschool thesis.

at or very near static level through the interval of pumping. The solution developed is for the steady-state condition, wherein it is assumed that the drawdown is zero at $r=\infty$.

Jacob (1946) also analyzed this problem, verifying the solution for steady flow and also developing a solution for the transient state. His final steady-state equation, in nondimensional form, for the drawdown in an infinite artesian aquifer has the form

$$s=\frac{Q}{2\pi T}K_0(x) \tag{35}$$

or, in the usual Survey units,

$$s=\frac{229\,QK_0(x)}{T}, \tag{36}$$

where

$$x=\frac{br}{a} \tag{37}$$

and

$a=\sqrt{T/S}$
$b=\sqrt{P'/m'S}$
T=coefficient of transmissibility of the artesian aquifer in gallons per day per foot,
P'=coefficient of vertical permeability of the semipermeable confining bed, in gallons per day per square foot,
S=coefficient of storage of the artesian aquifer,
Q=rate of withdrawal by the pumped well, in gallons per minute,
m'=thickness of the semipermeable confining bed, in feet,
r=distance from the pumped well to the observation well, in feet,
s=drawdown in the observation well, in feet.

The symbol $K_0(x)$ is a notation widely but not universally used to identify the modified Bessel function of the second kind of the zero order. In order to avoid any misunderstanding of its present usage it is identified as follows:

$$K_0(x)=-[0.5772+\log_e(x/2)]I_0(x) + (1/1!)^2(x/2)^2+(1/2!)^2(x/2)^4(1+1/2) + (1/3!)^2(x/2)^6(1+1/2+1/3+\ldots, \tag{38}$$

where

$$I_0(x)=1+(x/2)^2/(1!)^2+(x/2)^4/(2!)^2+(x/2)^6/(3!)^2+\ .\ .\ .\ . \quad (39)$$

The notation $I_0(x)$ is used to represent the modified Bessel function of the first kind of zero order. Values of the function $K_0(x)$ over the range of interest for most ground-water problems are given in table 4.

Equations 36 and 37 may be rewritten in the following form:

$$\log s=\log\left[\frac{229\,Q}{T}\right]+\log K_0(x) \quad (40)$$

$$\log r=\log\left[\frac{a}{b}\right]+\log x \quad (41)$$

The bracketed portions of equations 40 and 41 include all the terms that have been assumed constant in the derivation. It follows then that the variable s is related to r in the same manner that $K_0(x)$ is related to x. Thus the form of equations 40 and 41 once again suggests the same convenient method of graphical solution that has already been described for resolving the Theis formula. A type curve for use in solving equations 36 and 37 is prepared by plotting on logarithmic graph paper the values given in table 4. In figure 27 curve AA is in part a duplication of the lower part of curve BB and in part an extension of that curve into the next lower log cycle.

The solution of equations 36 and 37 thus requires plotting the field observations of s and r, at some particular time t, on logarithmic graph paper, using the same size of logarithmic scale adopted for the type curve. The data curve is superposed on the type curve, the coordinate axes of the two curves being held parallel, and translated to the position that represents the best fit of the field data to the type curve. When the match position is found, the amount of shift or translation from the s scale to the $K_0(x)$ scale is measured by the bracketed term of equation 40, and the translation between the r scale and the x scale is represented by the bracketed member of equation 41. An arbitrary point is selected on the data curve and the coordinates of this common point on both the data curve and the type curve are recorded. These coordinates, when substituted in equations 36 and 37, permit computation of the coefficient of transmissibility, T, of the artesian bed, and the value of x, which has inherent in it the coefficient of vertical permeability of the leaky confining bed.

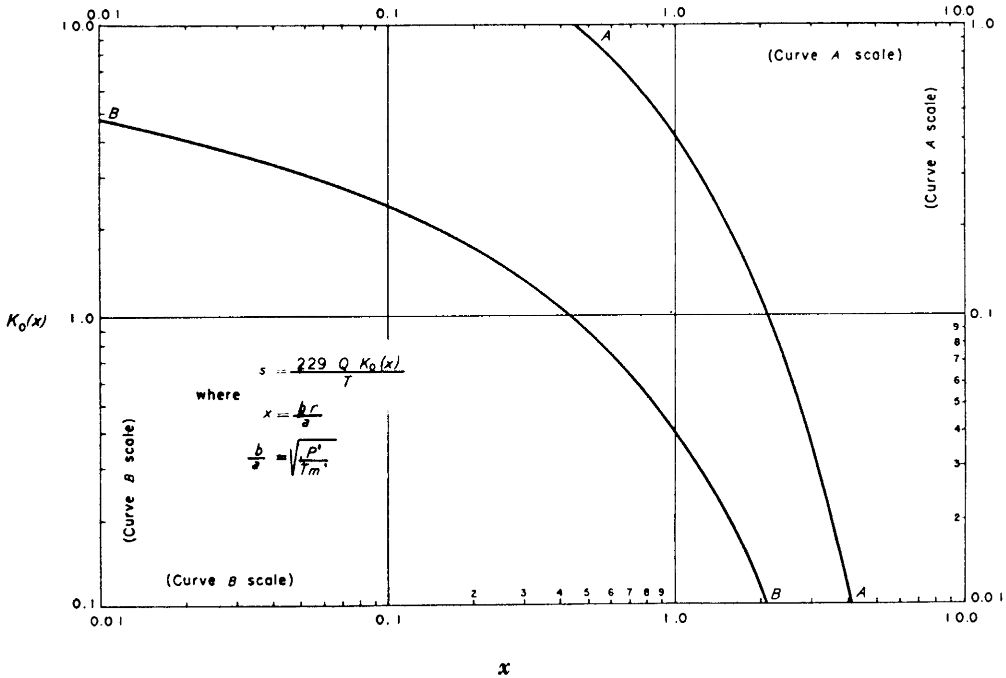


Figure 27.—Logarithmic graph of the modified Bessel function $K_0(x)$.

TABLE 4.—*Values of* $K_0(x)$, *the modified Bessel function of the second kind of zero order, for values of* x *between* 10^{-2} *and* 9.9

[Data for plotting type curve (fig. 27) used in solving equations 36 and 37. Values of $K_0(x)$ in the interval $0.01 \leq x \leq 1.00$ taken from tables in Commerce Dept. (1952, p. 36–60). Values of $K_0(x)$ in the interval $1.0 \leq x \leq 9.9$ taken from Gray, Mathews, and MacRobert (1931, p. 313–315)]

N	$x=N(10^0)$	$x=N(10^{-1})$	$x=N(10^{-2})$	N	$x=N(10^0)$	$x=N(10^{-1})$	$x=N(10^{-2})$
1.0	0.4210	2.4271	[1]4.7212	5.5	0.002139	.8466	
1.1	.3656			5.6	.001918		
1.2	.3185			5.7	.001721		
1.3	.2782			5.8	.001544		
1.4	.2437			5.9	.001386		
1.5	.2138	2.0300		6.0	.001244	.7775	2.9329
1.6	.1880			6.1	.001117		
1.7	.1655			6.2	.001003		
1.8	.1459			6.3	.0009001		
1.9	.1288			6.4	.0008083		
2.0	.1139	1.7527	4.0285	6.5	.0007259	.7159	
2.1	.1008			6.6	.0006520		
2.2	.08927			6.7	.0005857		
2.3	.07914			6.8	.0005262		
2.4	.07022			6.9	.0004728		
2.5	.06235	1.5415		7.0	.0004248	.6605	2.7798
2.6	.05540			7.1	.0003817		
2.7	.04926			7.2	.0003431		
2.8	.04382			7.3	.0003084		
2.9	.03901			7.4	.0002772		
3.0	.03474	1.3725	3.6235	7.5	.0002492	.6106	
3.1	.03095			7.6	.0002240		
3.2	.02759			7.7	.0002014		
3.3	.02461			7.8	.0001811		
3.4	.02196			7.9	.0001629		
3.5	.01960	1.2327		8.0	.0001465	.5653	2.6475
3.6	.01750			8.1	.0001317		
3.7	.01563			8.2	.0001185		
3.8	.01397			8.3	.0001066		
3.9	.01248			8.4	.00009588		
4.0	.01116	1.1145	3.3365	8.5	.00008626	.5242	
4.1	.009980			8.6	.00007761		
4.2	.008927			8.7	.00006983		
4.3	.007988			8.8	.00006283		
4.4	.007149			8.9	.00005654		
4.5	.006400	1.0129		9.0	.00005088	.4867	2.5310
4.6	.005730			9.1	.00004579		
4.7	.005132			9.2	.00004121		
4.8	.004597			9.3	.00003710		
4.9	.004119			9.4	.00003339		
5.0	.003691	.9244	3.1142	9.5	.00003006	.4524	
5.1	.003308			9.6	.00002706		
5.2	.002966			9.7	.00002436		
5.3	.002659			9.8	.00002193		
5.4	.002385			9.9	.00001975		

[1] When $x=0$, $K_0(x)=\infty$.

In application it is not possible to determine either a or b from field observation of steady flow, but their ratio can be determined from the definition of x:

$$x=r(b/a)=r\sqrt{\frac{P'/m'S}{T/S}}=r\sqrt{P'/Tm'}. \qquad (42)$$

The vertical permeability of the leaky bed can thus be determined from equation 42 if the bed thickness, m', is known. However, S, the coefficient of storage for the artesian aquifer cannot be determined as it is removed from the b/a ratio by cancellation. Hantush (1955) has designated the ratio P'/m' as the "leakage coefficient," and

Hantush and Jacob (1955) have described in considerable detail their development of equations for the nonsteady-state solution to the foregoing problem.

The preceding discussion has stipulated that equations 36, 37, and 42 are properly applied only to steady-state conditions. This means that enough time must have elapsed for the drawdown to have stabilized throughout the region for which the plot of s versus r is to be made. The manner in which the drawdown stabilizes at observation points at selected distances from the discharging well is shown on a semilogarithmic plot by Hantush and Jacob (1955, fig. 1). In effect their plot shows individual time-drawdown curves because values of drawdown divided by a constant are plotted against values of the logarithm of time multiplied by a constant. Of special interest is the fact that for all the curves, regardless of the represented distance from the discharging well, the drawdown stabilizes or levels off at the same value of time.

Assuming, therefore, that the requirement of stabilized drawdown has been met, an important feature of the logarithmic type curve (fig. 27) should be recognized. Note that the curve is drawn only for values of x greater than 0.01. Thus the matching of a logarithmic plot of s versus r against the leaky-aquifer type curve is appropriate only if the observed data and computed results can be shown to yield values of x (which is directly related to r) that are greater than 0.01. Actually the critical value of x is about 0.03, as can be demonstrated in the following manner.

In the tables of the Bessel functions (U.S. Department of Commerce, 1952) the following relation applies for small values of x:

$$K_0(x) = E_0(x) + F_0(x) \log_{10}(x).$$

The tables show that for values of x ranging from 0 to about 0.03 the values of $E_0(x)$ and $F_0(x)$ are 0.116 and -2.303 respectively. Substituting these equivalents in the above relation yields

$$K_0(x) = 0.116 - 2.303 \log_{10}(x),$$

which, by substitution from equation 37 and conversion to the natural logarithm, becomes

$$K_0(x) = 0.116 - \log_e(br/a).$$

If equation 35 is rewritten in terms of the difference in drawdown between two points at radii r_1 and r_2 (where $r_2 > r_1$) on the cone of depression, and the foregoing relation for $K_0(x)$ substituted therein, there follows the expression

$$s_1-s_2=\frac{Q}{2\pi T}\left[\left(0.116-\log_e \frac{br_1}{a}\right)-\left(0.116-\log_e \frac{br_2}{a}\right)\right],$$

or

$$s_1-s_2=\frac{Q}{2\pi T}\log_e \frac{r_2}{r_1},$$

which is the familiar Thiem equilibrium formula previously presented in the form of equation 1. The conclusion to be drawn is that in the region $x<0.03$ a logarithmic plot of s versus r exhibits only the effects of radial flow through the aquifer toward the discharging well; the leakage effects are not significant enough to influence the shape of the curve. Although the leaky-aquifer type curve could be extended readily into this region of low x values, its curvature is insensitive to leakage and is too slight to permit a matching that would be definitive of the x value needed for computing the leakage coefficient.

The nature of the abbreviated relation for $K_0(x)$, presented in the preceding discussion, suggests a simple means for analyzing the steady-state drawdown data within the region $x<0.03$. Rewriting equation 35 in terms of this special relation for $K_0(x)$ produces

$$s=\frac{Q}{2\pi T}\left[0.116-\log_e (br/a)\right],$$

or, in the usual Survey units and the common logarithm,

$$s=\frac{229Q}{T}\left(0.116-2.303 \log_{10} \frac{br}{a}\right). \qquad (43)$$

Recognizing that s and r are the only variables in equation 43, obviously a semilogarithmic plot of s versus $\log r$ produces a straight line. If r_0 is the intercept of this straight line at the zero-drawdown axis, appropriate substitution in equation 43 yields

$$\log_{10} \frac{br_0}{a}=\frac{0.116}{2.303},$$

or

$$\frac{b}{a}=\frac{1.12}{r_0}. \qquad (43a)$$

The analysis of steady-state test data for a leaky aquifer can thus be summarized in the following three simple procedures:

1. Select for plotting only the drawdown data which are within the region where drawdowns have levelled off.
2. Use equations 36, 37, and 42 with a logarithmic plot of s versus r, matched to the leaky-aquifer type curve (fig. 27), only if the observed data and resulting computations produce values of x greater than 0.03.
3. Use equations 2, 4, and 43a with a semilogarithmic plot of s versus log r if the data and resulting computations produce values of x less than 0.03.

The earliest observations of drawdown in each observation well, when s is small, should conform to the Theis nonequilibrium type curve for the infinite (nonleaky) aquifer if the rate of leakage from the confining bed is comparatively small. The coefficient of storage for the artesian aquifer can be determined under these conditions from the earliest observations of drawdown (Jacob, 1946, p. 204). The computed coefficient of transmissibility should be checked by comparing the value obtained from matching the earliest data to the nonequilibrium type curve with the value obtained by matching the later data to the steady-state leaky-aquifer type curve. If consistency of the T values is not obtained, then the leakage may be causing too much deviation at the smaller values of t to permit application of the Theis nonequilibrium formula.

VARIABLE DISCHARGE WITHOUT VERTICAL LEAKAGE

By R. W. Stallman

CONTINUOUSLY VARYING DISCHARGE

The rate at which water is pumped from a well or well field commonly varies with time in response to seasonal changes in demand. For instance, the pumping rate, as shown by records of daily or monthly discharge, is often found to be varying continuously. Where this element of variability is recognized in ground-water problems, the analytical methods that are described in the preceding sections of this report are not applicable without some modification or approximation. Exact equations could perhaps be developed for the case of continuously varying discharge, but the cost of analysis, in terms of time and effort, would likely be prohibitive considering that a separate and specific solution would be required for each problem. It is considered more expedient, therefore, to utilize the existing analytical methods, rendering them applicable to the field situation by introducing tolerable approximations of the field conditions. As an example, consider a situation where the pumping rate in a well (which may also represent a well field) tapping an artesian aquifer varies continuously with time in the manner indicated by the smooth curve shown in