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LINEAR ERROR ANALYSIS OF SLOPE-AREA DISCHARGE DETERMINATIONS

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ABSTRACT

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The slope-area method can be used to calculate peak flood discharges when current-meter measurements are not possible. This calculation depends on several quantities, such as water-surface fall, that are subject to large measurement errors. Other critical quantities, such as Manning's *n*, are not even amenable to direct measurement but can only be estimated. Finally, scour and fill may cause gross discrepancies between the observed condition of the channel and the hydraulic conditions during the flood peak.

The effects of these potential errors on the accuracy of the computed discharge have been estimated by statistical error analysis using a Taylor-series approximation of the discharge formula and the well-known formula for the variance of a sum of correlated random variates. The resultant error variance of the computed discharge is a weighted sum of covariances of the various observational errors. The weights depend on the hydraulic and geometric configuration of the channel.

The mathematical analysis confirms the rule of thumb that relative errors in computed discharge increase rapidly when velocity heads exceed the water-surface fall, when the flow field is expanding and when lateral velocity variation (alpha) is large. It also confirms the extreme importance of accurately assessing the presence of scour or fill.

INTRODUCTION

The slope-area method frequently is used to determine peak flood discharges when direct (current-meter) measurements cannot be obtained. The accuracy of slope-area determinations is of concern because such measurements often are the principal basis for extension of stage-discharge ratings into the flood-stage regime. Although examination of the hydraulic equations of the slope-area method does reveal important factors influencing the accuracy of the measurement, more precise quantitative error estimates are desired. This paper presents a statistical error analysis that shows how errors in the individual measured or estimated variables propagate through the slope-area equations to affect the computed discharge.

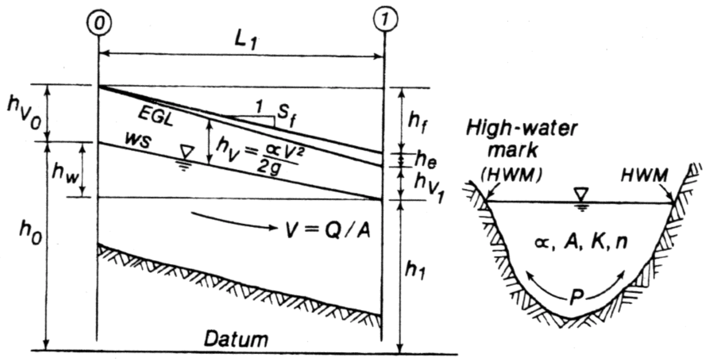


Fig. 1. Definition sketch for slope-area formula.

HYDRAULIC FORMULAS

The slope-area computations, like other applications of classical open-channel hydraulics, are based on the principles of conservation of matter (continuity equation) and conservation of energy (Bernoulli equation). Although the passage of a flood hydrograph involves unsteady flow, it is assumed that quasi-steady conditions prevail at the instant of occurrence of the peak flow at any point and that the high-water marks traced out by the flood crest define the water-surface elevations and slopes (Henderson, 1966, p. 376). The hydraulic principles and computational procedures of the slope-area method are described by Dalrymple and Benson (1967). General techniques of high-water mark identification and field data collection are described by Benson and Dalrymple (1967).

Figure 1 shows generalized profile and cross-section views of a slope-area channel reach. The channel is assumed to be nonuniform, so the water-surface profile, energy grade line and channel bed are nonparallel. The energy equation may be written between sections 0 and 1 as follows:

$$h_0 + h_{v_0} = h_1 + h_{v_1} + h_f + h_e \tag{1}$$

in which h_0 and h_1 are the elevations of the water surface at sections 0 and 1, h_{v_0} and h_{v_1} are the velocity heads, h_f is the head loss due to boundary friction and h_e is the head loss due to expansion or contraction of the flow field. The water-surface elevations h_0 and h_1 are defined by high-water marks observed in the field; the other terms can be expressed in terms of the (unknown) discharge Q.

The friction-loss term h_f is the central term in the slope-area computation. To evaluate this term, the U.S. Geological Survey uses Manning's formula as follows:

$$Q = \frac{c}{n} A R^{2/3} S_f^{1/2} = K S_f^{1/2} \tag{2}$$

in which c is a dimensional constant (equal to 1 when lengths are measured in meters and equal to 1.486 when lengths are in feet), n is Manning's roughness coefficient, A is cross-sectional area, R is hydraulic radius (A/P, ratio of area to wetted perimeter) and S_f is the friction slope (h_f/L_1, ratio of friction head loss to reach length L_1). $K = (c/n)AR^{2/3}$ is called the conveyance of the cross section. Methods for determining n-values on the basis of field observations are described in several sources (Chow, 1959; Benson and Dalrymple, 1967; Barnes, 1967; Arcement and Schneider, 1984).

Manning's formula was developed for so-called unit channels, of generally trapezoidal, rectangular, or semicircular cross-sectional shape, in which the entire cross section constitutes a single compact uniform-flow field. Natural stream channels, in contrast, often have compound cross sections with markedly variable flow depths and Manning's roughness values.

Manning's formula is applied to compound cross sections by subdividing them into unit subchannels or subareas. Subdivision is performed in such a way that each subarea is a compact shape with approximately uniform depth, roughness and flow velocity, and with negligible interactions with adjacent subareas. The conveyance of the compound channel is the sum of the subarea conveyances. Manning's formula for the compound section then is:

$$Q = (\Sigma K_i) S_f^{1/2} \tag{3}$$

in which the K_i are the subarea conveyances. Guidance on subdivision is provided by Benson and Dalrymple (1967) and by Davidian (1984).

In natural stream channels, conveyance commonly varies with distance along the channel. If the conveyance varies linearly between cross sections and if Manning's equation defines the friction slope at each intermediate point, then integration along the channel yields the result:

$$h_f = Q^2 L_1 / K_0 K_1 \tag{4}$$

in which K_0 and K_1 are the total conveyances of the cross sections at the ends of the reach. Thus, the effective conveyance of a reach with linearly varying conveyance is the geometric mean of the end-section conveyances.

The velocity head at a cross section is given by:

$$h_v = \alpha V^2/2g = \alpha (Q/A)^2/2g \tag{5}$$

in which $V = Q/A$ is the mean velocity in the cross section, g is the acceleration of gravity and α is a velocity-head coefficient that expresses the effect of cross-sectional nonuniformity on the kinetic energy flux. In the absence of site-specific information on velocity distributions, the Geological Survey assumes that $\alpha = 1$ for unsubdivided unit cross sections. For compound sections, an approximate velocity distribution is obtained by applying Manning's formula independently to each subarea, with the result:

$$\alpha = \sum (K_i^3/A_i^2)/(K^3/A^2) \tag{6}$$

in which the K_i are subarea conveyances.

Finally, energy losses caused by increased eddying and turbulence in expanding or contracting flows must be considered. These losses are poorly understood and there is no unanimity on how they should be computed (Henderson, 1966, p. 237; Benson and Dalrymple, 1967). The treatment of expansions is particularly questionable and such reaches should be avoided if possible when selecting slope-area measurement sites. The Geological Survey expresses these losses in terms of change in velocity head as:

$$h_e = k_1(h_{v_1} - h_{v_0}) \tag{7}$$

in which:

$$k_1 = \begin{cases} -k = -0.5, & \text{if } h_{v_1} < h_{v_0} \text{ (expanding)} \\ k' = 0, & \text{if } h_{v_1} > h_{v_0} \text{ (contracting)} \end{cases} \tag{8}$$

(Benson and Dalrymple, 1967). By referring to the definition of velocity head, the criterion for expanding and contracting reaches may be expressed in terms of the ratio $(A_1/\sqrt{\alpha_1})/(A_0/\sqrt{\alpha_0})$, which is greater than 1 for expanding reaches and less than 1 for contracting ones.

The energy equation now may be written in terms of the water-surface fall and the (unknown) discharge as follows:

$$h_0 - h_1 = \frac{Q^2 L_1}{K_0 K_1} + (1 + k_1)\frac{Q^2}{2g}\left(\frac{\alpha_1}{A_1^2} - \frac{\alpha_0}{A_0^2}\right) \tag{9}$$

in which K_0 and K_1 are the (total) conveyances of cross sections 0 and 1. In practice, several cross sections are used to get an improved representation of the geometry and hydraulics of the reach. Equation (9) can be applied to each of the subreaches $i = 1, 2, \ldots, M$. The discharge Q is the same in each subreach. The total water-surface fall h_w is the sum of the subreach falls. The resulting equation can be solved for Q as follows:

$$Q = \sqrt{h_w}\left[\sum L_i/K_i K_{i-1} + \sum(1 + k_i)\left(\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right)\Big/ 2g\right]^{-1/2} \tag{10}$$

in which K_i and K_{i-1} are the (total) conveyances of cross sections i and $i - 1$. The first summation on the right-hand side of this equation represents the effects of boundary friction (Manning's equation) whereas the second summation represents all of the velocity-variation (nonuniformity) effects. The slope-area method is intended for use when the friction term is the dominant one.

SOURCES OF ERROR

Applications of the slope-area method are subject to two general types of errors: observational errors and theoretical errors. Observational errors include all discrepancies between the true values of all variables and the corresponding measured, estimated, or assumed values actually used in the computation. Such errors include surveying errors, errors in determining water-surface profile slopes from scattered and inconsistent high-water marks, and misestimates of Manning's n. Theoretical errors include effects of all applications of the slope-area model to flows that are not governed by that model. Such flows include mudflows (as opposed to water floods) and flows with excessive expansions or contractions. Jarrett (1987, this volume) has described and evaluated a number of potential theoretical and observational errors in slope-area applications. This paper considers only the effects of observational errors. Any error estimates resulting from this study therefore must be considered as lower bounds, predicted on a proper correspondence between the slope-area theory and the hydraulic conditions in the field.

ERROR ANALYSIS OF DISCHARGE FORMULA

For purposes of error analysis, it is convenient to use eqn. (10) to express the discharge Q as a function of $3 + 4(M + 1) + M$ variables, as follows:

$$Q = f(h_w, k, k', \alpha_0, \ldots, \alpha_M, A_0, \ldots, A_M, P_0, \ldots, P_M, n_0, \ldots, n_M, L_1, \ldots, L_M) \tag{11}$$

in which the conveyances K_i have been expressed in terms of the corresponding A_i, P_i and n_i. When $M = 2$ (three-section formula), Q is a function of 17 variables. It is assumed that the mathematical form of the function f is correct (no theoretical error) and that the only errors are observational. If the errors are small enough, their effects on Q can be adequately represented by a first-order Taylor series with coefficients $\partial Q/\partial x$ evaluated at the observed values of the independent variables. Then the variance of the discharge error ΔQ can be expressed in terms of the $\partial Q/\partial x$ and the variances and covariances of the independent variables (Benjamin and Cornell, 1970, p. 180ff). Fortunately, most of the covariance terms are zero. The result is as follows:

$$\begin{aligned}
\operatorname{var} \Delta Q = {} & \left(\frac{\partial Q}{\partial h_w}\right)^2 \operatorname{var} \Delta h_w + \left(\frac{\partial Q}{\partial k}\right)^2 \operatorname{var} \Delta k \\
& + \left(\frac{\partial Q}{\partial k'}\right)^2 \operatorname{var} \Delta k' + \sum_{i=0}^{M} \left(\frac{\partial Q}{\partial \alpha_i}\right)^2 \operatorname{var} \Delta \alpha_i \\
& + \sum_{i=0}^{M} \left(\frac{\partial Q}{\partial A_i}\right)^2 \operatorname{var} \Delta A_i + \sum_{i=0}^{M} \left(\frac{\partial Q}{\partial P_i}\right)^2 \operatorname{var} \Delta P_i \\
& + \sum_{i=0}^{M} \left(\frac{\partial Q}{\partial n_i}\right)^2 \operatorname{var} \Delta n_i + \sum_{i=1}^{M} \left(\frac{\partial Q}{\partial L_i}\right)^2 \operatorname{var} \Delta L_i \\
& + 2 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \frac{\partial Q}{\partial n_i} \frac{\partial Q}{\partial n_j} \operatorname{cov}(\Delta n_i, \Delta n_j) \\
& + 2 \sum_{i=0}^{M-1} \sum_{j=i+1}^{M} \frac{\partial Q}{\partial A_i} \frac{\partial Q}{\partial A_j} \operatorname{cov}(\Delta A_i, \Delta A_j) \\
& + 2 \sum_{i=0}^{M} \frac{\partial Q}{\partial P_i} \frac{\partial Q}{\partial A_i} \operatorname{cov}(\Delta A_i, \Delta P_i)
\end{aligned} \tag{12}$$

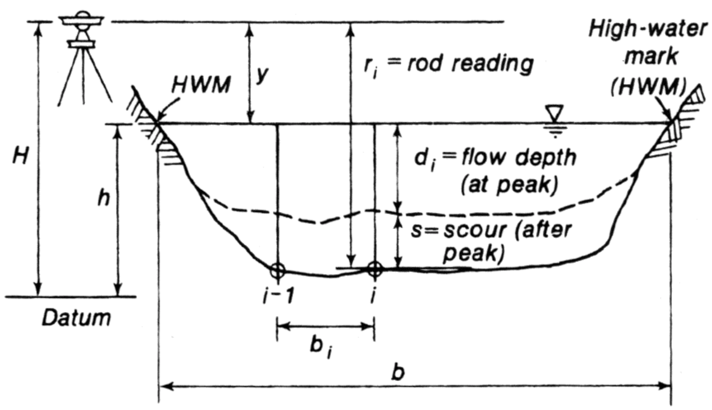


Fig. 2. Definition sketch for cross-section geometry.

The partial derivatives and the variances and covariances of the input-variable errors now have to be determined.

ERROR ANALYSIS OF CROSS-SECTION PROPERTIES

Figure 2 defines the various measurements and relationships that are used to determine the geometry of slope-area cross sections. Scour and fill are sources of potentially severe errors (Benson and Dalrymple, 1967). In this analysis it is assumed that scour or fill might have occurred after the high-water marks were made and that the effect is to make the flow depth manifested by post-flood evidence different from the true flow depth at the time of the peak flow. The scour (if any occurs) is assumed to be of constant depth throughout any one cross section but to vary from section to section; scour depths may be correlated between sections (Jarrett, 1987, this volume). Fill is treated as negative scour.

Under these assumptions, the flow depth at any point in the cross section is $d_i = r_i - y - s$, where r_i is a rod reading, y is an equivalent measurement down to the water surface (as defined by high-water marks) and s is the scour depth. The cross-sectional area and wetted perimeter are:

$$A = \sum_{i=1}^{N} b_i(r_i + r_{i-1})/2 - by - bs \qquad (13)$$

$$P = \sum_{i=1}^{N} [b_i^2 + (r_i - r_{i-1})^2]^{1/2} \qquad (14)$$

where b_i is the transverse distance between ground points $i - 1$ and i, $N + 1$ is the number of ground points and $b = \Sigma b_i$ is the total top width.

The effects of measurement errors on the computed cross-sectional area can be determined as follows:

$$\text{var}\,\Delta A = \sum_{i=1}^{N}\left(\frac{\partial A}{\partial b_i}\right)^2 \text{var}\,\Delta b_i + \sum_{i=1}^{N}\left(\frac{\partial A}{\partial r_i}\right)^2 \text{var}\,\Delta r_i + \left(\frac{\partial A}{\partial y}\right)^2 \text{var}\,\Delta y + \left(\frac{\partial A}{\partial s}\right)^2 \text{var}\,\Delta s \tag{15}$$

where it is assumed that the b, r, y and s measurement errors are statistically uncorrelated. The partial derivatives are:

$$\begin{aligned} \partial A/\partial b_i &= (r_i + r_{i-1})/2 - y - s \cong \bar{d} = A/b \\ \partial A/\partial r_i &= (b_i + b_{i-1})/2 \cong b/N \\ \partial A/\partial y &= \partial A/\partial s = -b \end{aligned} \tag{16}$$

Note that it has been assumed that the ground points are approximately equally spaced and that the depth at all points is approximately equal to the mean depth. This kind of uniformity assumption will be made throughout the remainder of the paper in order to make the error equations more readily understandable. For the same reason, measurement errors in r_i, y and s are expressed in terms of relative errors ε_r, ε_h and ε_s referenced to mean depth $\bar{d} = A/b$. The error in b_i is expressed in terms of relative error ε_b referenced to b/N, the average spacing of ground points. Values of ε_r and ε_h are distinguished because the measurements of ground points and of high-water marks are done with different techniques and different accuracy. Making the indicated substitutions yields:

$$\varepsilon_A^2 = \frac{\text{var}\,\Delta A}{A^2} = \frac{\varepsilon_b^2}{N} + \frac{\varepsilon_r^2}{N} + \varepsilon_h^2 + \varepsilon_s^2 \tag{17}$$

in which ε_A is the relative standard error of the cross-sectional area.

The covariance of area errors at cross sections j and m can be shown to be:

$$\text{cov}(\Delta A_j, \Delta A_m) = \frac{\partial A_j}{\partial S_j}\frac{\partial A_m}{\partial S_m}\,\text{cov}(\Delta S_j, \Delta S_m) \tag{18}$$

where it is assumed that all geometric variables except scour depth are independent between cross sections. It is further assumed that scour consists of a (random) reach-wide general scour plus independent random variations from section to section. Thus the scour error variance at each section is var ΔS_j = var $\Delta S'$ + var $\Delta S''$, where $\Delta S'$ denotes the reach-wide general scour error and $\Delta S''$ denotes the independent random component, which is identically distributed at all sections. It can be shown that $\text{cov}(\Delta S_j, \Delta S_m) = \text{var}\,\Delta S'$. In terms of correlation coefficients:

$$\rho_{\Delta s} = \frac{\text{cov}(\Delta S_j, \Delta S_m)}{\sqrt{\text{var}\,\Delta S_j}\sqrt{\text{var}\,\Delta S_m}} = \frac{\text{var}\,\Delta S'}{\text{var}\,\Delta S' + \text{var}\,\Delta S''} \tag{19}$$

which is constant for all cross-section pairs. Substituting partial derivatives and error estimates yields:

$$\mathrm{cov}(\Delta A_j, \Delta A_m) = b^2 \rho_{\Delta s} \varepsilon_s^2 \bar{d}^2 = \rho_{\Delta s} A^2 \varepsilon_s^2 \qquad (20)$$

in which $\rho_{\Delta s}$ is the assumed section-to-section correlation between scour-depth errors and $\varepsilon_s^2 = \mathrm{var}\, \Delta S/\bar{d}$. Channel widths, depths and areas have been assumed sufficiently uniform from section to section to be approximated by a single value.

In similar fashion, the errors in wetted perimeter can be shown to have:

$$\mathrm{var}\, \Delta P = \sum_{i=1}^{N} \left(\frac{\partial P}{\partial b_i}\right)^2 \mathrm{var}\, \Delta b_i + \sum_{i=0}^{N} \left(\frac{\partial P}{\partial r_i}\right)^2 \mathrm{var}\, \Delta r_i \qquad (21)$$

$$\mathrm{cov}(\Delta P, \Delta A) = \sum_{i=1}^{N} \frac{\partial A}{\partial b_i} \frac{\partial P}{\partial b_i} \mathrm{var}\, \Delta b_i + \sum_{i=0}^{N} \frac{\partial A}{\partial r_i} \frac{\partial P}{\partial r_i} \mathrm{var}\, \Delta r_i \qquad (22)$$

It is assumed that the channel cross section is relatively wide, shallow and uniform. Then the wetted perimeter approximately equals the top width ($P \cong B$), $\partial P/\partial b_i \cong 1$ and $\partial P/\partial r_i \cong 0$. Carrying out the indicated substitutions:

$$\mathrm{var}\, \Delta P/P^2 = \varepsilon_P^2 \cong \varepsilon_b^2/N \qquad (23)$$

$$\mathrm{cov}(\Delta P, \Delta A)/AP \cong \varepsilon_P^2 \cong \varepsilon_b^2/N \qquad (24)$$

PARTIAL DERIVATIVES OF THE DISCHARGE FORMULA

As noted above, the slope-area discharge formula is a function of 3 + 4 (M + 1) + M variables, where M is the number of subreaches. For $M = 2$ ("3-section formula") there are 17 variables. The partial derivatives of Q with respect to all these variables have to be evaluated and properly summed. This task is mathematically straightforward but clerically cumbersome.

To keep the work under control, additional notation is needed. Define, for subreaches $i = 1, \ldots, M$:

$$G_i = \frac{L_i}{K_i K_{i-1}} = \frac{L_i n_i n_{i-1} P_i^{2/3} P_{i-1}^{2/3}}{c^2 A_i^{5/3} A_{i-1}^{5/3}} \qquad (25)$$

$$H_i = (1 + k_i) \frac{1}{2g} \left(\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right) \qquad (26)$$

The discharge eqn. (10) then becomes:

$$Q = \sqrt{h_w} \left(\sum_{i=1}^{M} G_i + \sum_{i=1}^{M} H_i\right)^{-1/2} \qquad (27)$$

The derivative of Q with respect to h_w now can be seen to be:

$$\frac{\partial Q}{\partial h_w} = \frac{1}{2} \frac{Q}{h_w} \qquad (28)$$

For the other variables:

$$\frac{\partial Q}{\partial x} = -\frac{1}{2}\frac{Q^3}{h_w}\left(\frac{\partial \Sigma G_i}{\partial x} + \frac{\partial \Sigma H_i}{\partial x}\right) \tag{29}$$

Many of the derivatives involve terms of the form $(G_i + G_{i+1})$; others involve only G_1 or only G_M. To enable all of these terms to be represented in a uniform fashion, define:

$$G_0 = G_{M+1} = H_0 = H_{M+1} = 0;\ k_0 = k_{M+1} = -1 \tag{30}$$

Finally, to improve the understandability of the error equations, let each independent variable that varies from section to section be approximated by a single value that is constant throughout the reach. (This approximation is used only in evaluating the errors, not in calculating the discharge.) Under this approximation, terms of the form $(G_i + G_{i+1})$ can be expressed as $G(u_i + u_{i+1})$, where G is constant for all sections and u_i is defined by:

$$u_0 = u_{M+1} = 0;\ u_1 = \ldots = u_M = 1 \tag{31}$$

To illustrate the procedure, the derivative of Q with respect to the area of section i can be computed as follows:

$$\frac{\partial Q}{\partial A_i} = \frac{1}{2}\frac{Q^3}{h_w}\frac{1}{A_i}\left[\frac{5}{3}(G_i + G_{i+1}) + 2(k_i - k_{i+1})\frac{\alpha_i}{2g\,A_i^2}\right]$$

Note that $Q^2 G_i = h_{f_i}$ is the friction loss in subreach i and that $Q^2\alpha_i/2gA_i^2 = h_{v_i}$ is the velocity head. Under the uniformity approximation, $h_{f_i} \cong h_f/M$ and $h_{v_i} \cong h_v$. Then:

$$\frac{\partial Q}{\partial A_i} = \frac{1}{2}\frac{Q}{A_i}\left[\frac{5}{3}\frac{1}{M}\frac{h_f}{h_w}(u_i + u_{i+1}) + 2(k_i - k_{i+1})\frac{h_v}{h_w}\right] \tag{32}$$

where the u_i terms are needed for proper handling of the cases $i = 0$ and $i = M$.

In similar fashion the other partial derivatives can be evaluated as follows:

$$\frac{\partial Q}{\partial L_i} = -\frac{1}{2}\frac{Q}{L_i}\frac{1}{M}\frac{h_f}{h_w}u_i \tag{33}$$

$$\frac{\partial Q}{\partial n_i} = -\frac{1}{2}\frac{Q}{n_i}\frac{1}{M}\frac{h_f}{h_w}(u_i + u_{i+1}) \tag{34}$$

$$\frac{\partial Q}{\partial P_i} = -\frac{1}{2}\frac{2}{3}\frac{Q}{P_i}\frac{1}{M}\frac{h_f}{h_w}(u_i + u_{i+1}) \tag{35}$$

$$\frac{\partial Q}{\partial \alpha_i} = -\frac{1}{2}\frac{Q}{\alpha_i}(k_i - k_{i+1})\frac{h_v}{h_w} \tag{36}$$

$$\frac{\partial Q}{\partial k} = \frac{1}{2}Q\sum\Delta h_v^- / h_w \tag{37}$$

$$\frac{\partial Q}{\partial k'} = -\frac{1}{2} Q \sum \Delta h_v^+ / h_w \tag{38}$$

where $\Sigma\Delta h_v^-$ represents the sum of negative velocity-head changes:

$$\Sigma\Delta h_v^- = \sum_{i=1}^{M} \frac{Q^2}{2g}\left[\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right]^- \tag{39}$$

$$[x]^- = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geqslant 0 \end{cases} \tag{40}$$

and $\Sigma\Delta h_v^+$ represents the similar sum of positive velocity-head changes.

EVALUATION OF THE DISCHARGE ERROR

The variance of the discharge error now can be computed by substituting the partial derivatives of Q and the variances and covariances of the measurement errors into eqn. (12). The error in estimation of Manning's n is assumed to follow a uniform-correlation model, similar to that for scour, eqn. (19). Thus, $\mathrm{cov}(\Delta n_i, \Delta n_j) = \rho_{\Delta n}\varepsilon_n^2 n_i n_j$, where ε_n is the relative standard error of Manning's n and $\rho_{\Delta n}$ is the correlation between n-errors at different cross sections. Variances and covariances of area and wetted-perimeter errors are given in eqns. (17)–(24). Making the indicated substitutions and carrying out straightforward though laborious algebraic manipulations yields the final result:

$$\begin{aligned}
\frac{\mathrm{var}\,\Delta Q}{Q^2} = {} & \frac{1}{4}\varepsilon_{h_w}^2 + \frac{1}{4}\varepsilon_L^2\frac{\phi^2}{M} \\
& + \left(\frac{\varepsilon_r^2}{N} + \varepsilon_h^2\right)\left[\left(\frac{5}{3}\phi\right)^2 \frac{1 - \dfrac{1}{2M}}{M} - \frac{5}{3}\phi\psi\frac{k_M - k_1}{M}\right. \\
& \left. + \psi^2 \sum_{i=0}^{M} (k_i - k_{i+1})^2\right] \\
& + \frac{\varepsilon_{b2}}{N}\left[\phi^2 \frac{1 - \dfrac{1}{2M}}{M} - \phi\psi\frac{k_M - k_1}{M} + \psi^2 \sum_{i=0}^{M} (k_i - k_{i+1})^2\right] \\
& + \mathrm{var}\,\Delta k\,\frac{1}{4}\psi^2(\Sigma\Delta h_v^- / h_v)^2 + \mathrm{var}\,\Delta k'\,\frac{1}{4}\psi^2(\Sigma\Delta h_v^+ / h_v)^2 \\
& + \varepsilon_\alpha^2\frac{1}{4}\psi^2\sum_{i=0}^{M}(k_i - k_{i+1})^2 + \varepsilon_n^2\phi^2\left[\rho_{\Delta n} + (1 - \rho_{\Delta n})\frac{1 - \dfrac{1}{2M}}{M}\right]
\end{aligned}$$

$$+\ \varepsilon_s^2 \left\{ \rho_{\Delta s} \left(\frac{5}{3}\phi\right)^2 + (1 - \rho_{\Delta s}) \left[\left(\frac{5}{3}\phi\right)^2 \frac{1 - \dfrac{1}{2M}}{M} - \frac{5}{3}\phi\psi\,\frac{k_M - k_1}{M} \right.\right.$$

$$\left.\left. +\ \psi^2 \sum_{i=0}^{M} (k_i - k_{i+1})^2 \right]\right\} \tag{41}$$

in which:

$$\phi = h_f/h_w \tag{42}$$

$$\psi = h_v/h_w \tag{43}$$

This formula represents simply a weighted sum of the various measurement-error variances, with the weights depending on the geometric and hydraulic configuration of the reach. For convenience of reference, it may be written in the form:

$$\varepsilon_Q^2 = \frac{\mathrm{var}\ \Delta Q}{Q^2} = \sum_x \varepsilon_x^2 c_x \tag{44}$$

in which the summation runs over all error sources x, and in which it is understood that:

$$\varepsilon_k = \mathrm{var}\ \Delta k \tag{45}$$

$$\varepsilon_{k'} = \mathrm{var}\ \Delta k' \tag{46}$$

In addition, it will be convenient to write:

$$\varepsilon_{\bar{d}}^2 = \varepsilon_h^2 + \varepsilon_r^2/N \tag{47}$$

in which $\varepsilon_{\bar{d}}$ is the relative standard error of the hydraulic mean depth.

Inspection of the terms of eqn. (41) indicates that discharge error variance increases rapidly as ψ (ratio of velocity head to water-surface fall) and ϕ (ratio of friction head to water-surface fall) become large. The terms involving $\Sigma\Delta h_v^+$, $\Sigma\Delta h_v^-$, and the $\Sigma\Delta k^2$ all increase, and cause increasing discharge errors, as longitudinal nonuniformity increases. These results confirm the rules of thumb (Dalrymple and Benson, 1967) that relative errors in computed discharge increase rapidly when the velocity head exceeds the water-surface fall, when the friction loss exceeds the water-surface fall (expanding reaches) and when lateral velocity variation (alpha) is large.

NUMERICAL EXAMPLES

To illustrate the use of the above results, two numerical examples are given. The geometric and hydraulic details of these examples are taken from actual measurements described by Barnes (1967). The various measurement-error estimates, on the other hand, are purely hypothetical, so the computed discharge error variances should not be interpreted as evaluations of the specific

TABLE 1

Approximate hydraulic and geometric properties of sample reaches (after Barnes, 1967)

	Columbia (p. 30)	Oconee (P. 106)
n, Manning's n	0.030	0.041
Q, peak discharge ($m^3 s^{-1}$)	28,300	90
M, number of subreaches	3	4
L, reach length (m)	1,280	475
h_w, water-surface fall (m)	0.338	0.247
A, area (m^2)	8,350	90
b, width (m)	490	35
$\bar{d}$, mean depth (m)	17.4	2.6
R, hydraulic radius (m)	16.8	2.4
V, mean velocity (m/s)	3.35	0.98
h_v, velocity head (m)	0.572	0.049
$h_{v_M} - h_{v_0}$ (m)	+0.034	+0.007
h_e, expansion loss (m)	0.017	0.003
h_f, friction head (m)	0.286	0.237
$\phi = h_f/h_w$	0.85	0.96
$\psi = h_v/h_w$	1.70	0.20
$k_M - k_1$	−0.5	0.0
$\Sigma(k_i - k_{i+1})^2$	1.5	2.5
$\Sigma \Delta h_v^+ / h_v$	0.12	0.26
$\Sigma \Delta h_v^- / h_v$	−0.06	−0.12

TABLE 2

Evaluation of error-variance components

		Columbia		Oconee	
x	ε_x	c_x	$c_x \varepsilon_x^2$ (10^{-4})	c_x	$c_x \varepsilon_x^2$ (10^{-4})
h_w	0.10	0.25	25.0	0.25	25.0
$\bar{d}$	0.02	5.29	21.1	0.661	2.6
b	0.01	0.48	0.5	0.03	0.03
L	0.01	0.06	0.06	0.06	0.06
k'	0.10	0.011	1.1	0.0007	0.07
k	0.10	0.003	0.3	0.0001	0.01
α	0.10	1.10	110	0.025	2.5
n	0.10	0.45	45	0.562	56.2
$\rho_{\Delta n}$	0.50				
s	0.10	3.66	366	1.610	161.0
$\rho_{\Delta s}$	0.50				
(var ΔQ)/Q (10^{-4})			569		247
(std. dev. ΔQ)/Q (%)			24		16

TABLE 3

Sensitivity analysis

x	ε_x	ε_Q (%)	
		Columbia	Oconee
–	–	24	16
h_w	0.05	23	15
α	0.05	22	15
n	0.05	23	14
(α, n)	(0.05, 0.05)	21	14
s	0.05	17	11
$M + 1$	–	24	16
$\rho_{\Delta n} = 0$	–	23	15
$\rho_{\Delta s} = 1$	–	20	18
$\rho_{\Delta s} = 0$	–	27	12

measurements; the results are presented only to illustrate the method of computation and the influences of various factors.

The two examples presented are the flood of May 31, 1948, on the Columbia River at the Dalles, Oregon (Barnes, 1967, p. 30), and the flood of May 27, 1959, on the Middle Oconee River, near Athens, Georgia (Barnes, 1967, p. 106). The general geometric and hydraulic properties of the reaches are summarized in Table 1. The Columbia has much the higher velocities and the Middle Oconee has the greater predominance of friction loss. Both reaches are well within the subcritical flow regime.

Table 2 shows the various error sources and their contributions to the total error-variance of the computed discharge. The numerical values of the standard errors in the ε column are intended only as reasonable hypothetical values; they are not intended as specific evaluations of the technique or accuracy of the two example measurements. The coefficients c_x depend on the geometric and hydraulic configuration of the reach. They can be calculated from eqn. (41) and the information given in Table 1. Note the substantial differences in the coefficients for $\bar{d}$, b and α. These differences seem to be due primarily to the large differences in mean velocity and to a lesser extent to differences in degree of nonuniformity. In both examples the channels are reasonably uniform, so the influences of the uncertainties in the expansion- and contraction-loss coefficients are minimal. The dominance of the scour term in both examples is significant.

The influence coefficient c_x having been computed, a sensitivity analysis on the error estimates can be performed. Table 3 shows the results of changing one error-factor at a time in eqn. (41) while leaving all other terms at the values stated in Table 2. The line labeled $M + 1$ was computed by interpolating one additional hypothetical cross section into each of the measurement reaches. This result indicates that there is little benefit in interpolation additional cross sections into a uniform reach solely to increase the value of M. This analysis is unable, however, to determine the consequences of omitting cross sections that might be necessary to define the flow geometry of the reach, so it would be prudent to use as many cross sections as appeared necessary. Finally, it appears that the most significant improvements in discharge accuracy can be obtained by reducing uncertainty in the scour term; as long as scour uncertainty exists, improvements in measuring other factors seem to be of limited benefit.

REFERENCES

Arcement, G.J., Jr. and Schneider, V.R., 1984. Guide for selecting Manning's roughness coefficients for natural channels and flood plains. Rep. FHWA-TS-84-204, Nat. Tech. Inf. Serv., Springfield, Va., 62 pp.

Barnes, H.H., Jr., 1967. Roughness characteristics of natural channels. U.S. Geol. Surv. Water Supply Pap. 1849: 213 pp.

Benjamin, J.R. and Cornell, C.A., 1970. Probability, Statistics, and Decision for Civil Engineers. McGraw-Hill, New York, N.Y., 684 pp.

Benson, M.A. and Dalrymple, T., 1967. General field and office procedures for indirect discharge measurements. U.S. Geol. Surv. Techniques of Water-Resources Investigations. Book 3, Chapter A1, 30 pp.

Chow, V.T., 1959. Open-Channel Hydraulics. McGraw-Hill, New York, N.Y., 680 pp.

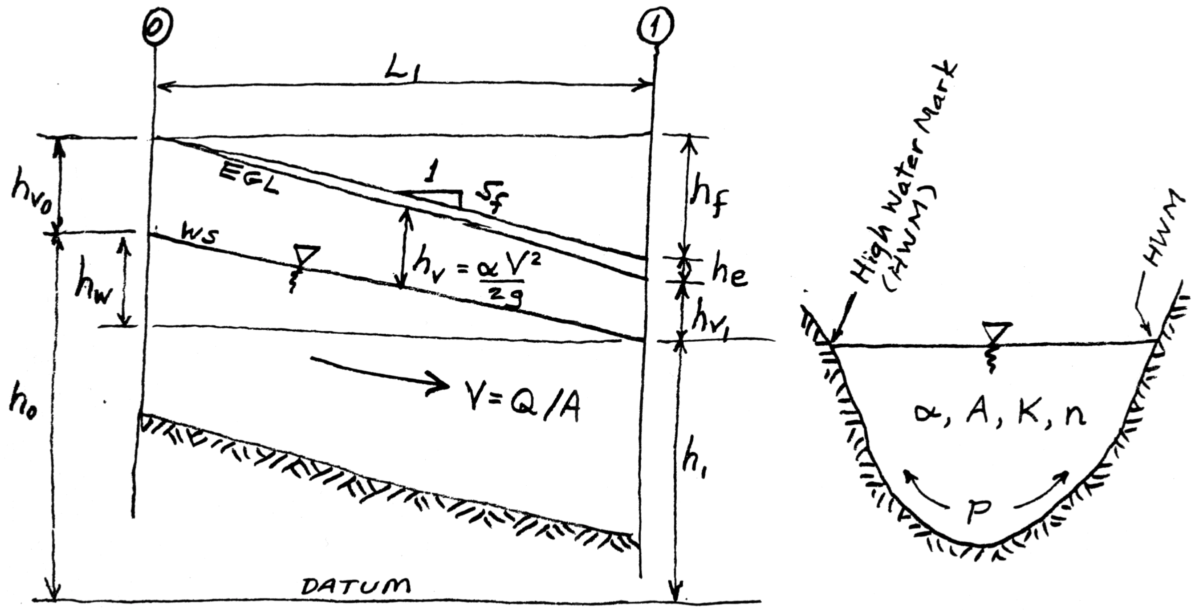
Dalrymple, T. and Benson, M.A., 1967. Measurement of peak discharge by the slope-area method. U.S. Geol. Surv. Techniques of Water-Resources Investigations. Book 3, Chapter A2, 12 pp.

Davidian, Jacob, 1984. Computation of water-surface profiles in open channels. U.S. Geol. Surv. Techniques of Water-Resources Investigations. Book 3, Chapter A15, 48 pp.

Henderson, F.M., 1966. Open Channel Flow. Macmillan, New York, N.Y., 522 pp.

Jarrett, R.D., 1987. Errors in slope-area computations of peak discharges in mountain streams. In: W.H. Kirby, S.-Q. Hua and L.R. Beard (Editors), Analysis of Extraordinary Flood Events. J. Hydrol., 96: 53–67 (this volume).

Slope-Area Method -- Definitions



Bernoulli Equation (Energy)

$$h_0 + h_{v0} = h_1 + h_{v1} + h_f + h_e$$

Velocity head -- $h_v = \alpha V^2/2g = \alpha Q^2/2gA^2$

Friction loss -- Manning's equation: $V = \frac{1}{n}(A/P)^{2/3} S^{1/2}$

Define conveyance: $K = \frac{1}{n} A (A/P)^{2/3}$

$$Q = \sqrt{K_0 K_1} \cdot \sqrt{S_f}$$

$$h_f = Q^2 L_1 / K_0 K_1$$

Eddy loss -- $h_e = k_1 \cdot \Delta h_v = k_1 \cdot (h_{v1} - h_{v0})$

$$k_1 = \begin{cases} k' & \text{if } h_{v1} \geq h_{v0} \left(\frac{A_1}{\sqrt{\alpha_1}} \leq \frac{A_0}{\sqrt{\alpha_0}} \right) \\ -k & \text{if } h_{v1} < h_{v0} \left(\frac{A_1}{\sqrt{\alpha_1}} > \frac{A_0}{\sqrt{\alpha_0}} \right) \end{cases}$$

(expanding reach)

$(k' \cong 0,\ k \cong 1/2)$

Slope-Area Method -- M-Section Solution

Bernoulli equation -- subreach i

$$h_{w_i} = h_{i-1} - h_i = h_{f_i} + h_{v_i} - h_{v_{i-1}} + he_i$$

$$= \frac{Q^2 L_i}{K_i K_{i-1}} + (1+k_i)\frac{Q^2}{2g}\left(\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right)$$

Summation over subreaches $i = 1, \ldots, M$
and solution for Q:

$$Q = \sqrt{\frac{h_0 - h_M}{\sum_{i=1}^{M} \frac{L_i}{K_i K_{i-1}} + \sum_{i=1}^{M} (1+k_i)\frac{1}{2g}\left(\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right)}}$$

Define:

$$L = \sum_{i=1}^{M} L_i \qquad h_w = h_0 - h_M = \sum_{i=1}^{M} h_{w_i}$$

$$\bar{K} = \sqrt{\frac{L}{\sum_{i=1}^{M} \frac{L_i}{K_i K_{i-1}}}}$$

$$\hat{Q} = \bar{K}\sqrt{h_w / L}$$

$$\hat{h}_{v_i} = \alpha_i \hat{Q}^2 / 2g A_i^2$$

Then:

$$Q = \hat{Q} \cdot C_w = \hat{Q}\sqrt{\frac{1}{1 + \sum_{i=1}^{M} (1+k_i)\left(\frac{\hat{h}_{v_i} - \hat{h}_{v_{i-1}}}{h_w}\right)}}$$

Also note:

$$h_{v_i} = (C_w)^2 \cdot \hat{h}_{v_i}$$

Slope-Area Method -- Error Sources

$$Q = \sqrt{\dfrac{h_w}{\sum \dfrac{L_i}{K_i K_{i-1}} + \sum (1+k_i)\dfrac{1}{2g}\left(\dfrac{\alpha_i}{A_i^2} - \dfrac{\alpha_{i-1}}{A_{i-1}^2}\right)}}$$

$$K = \frac{A^{5/3}}{n P^{2/3}} \qquad k_i = \begin{cases} -k & \text{if } \dfrac{\alpha_i}{A_i^2} < \dfrac{\alpha_{i-1}}{A_{i-1}^2} \\ k' & \text{if } \dfrac{\alpha_i}{A_i^2} > \dfrac{\alpha_{i-1}}{A_{i-1}^2} \end{cases}$$

---

h_w -- location of peak-flow static water surface relative to high-water marks; unsteady-flow loop-rating effects; inconsistent high-water marks

n -- subdivision of compound cross sections; heavy sediment loads; highly agitated flows in steep streams; channel irregularities

k, k' -- eddy losses poorly understood

α -- actual velocity distributions may differ from assumed distribution; little basis for estimating α from post-flood information

A -- scour and fill; faulty cross-section location

$K_i K_{i-1}$ -- nonlinear variation of conveyance between cross sections $i-1$ and i

First-Order Error Analysis -- General Theory

$$W = f(X_1, \ldots, X_n) \qquad \text{(measured)}$$
$$W^* = f(X_1^*, \ldots, X_n^*) \qquad \text{(true)}$$

Errors:

$$\Delta W = W - W^* = \sum \frac{\partial f}{\partial X_i}(X_i - X_i^*) + o(\Delta X)$$
$$= \sum \frac{\partial f}{\partial X_i} \cdot \Delta X_i$$

Expected Error:

$$E\Delta W = \sum \frac{\partial f}{\partial X_i} E\Delta X_i$$

Variance of Errors:

$$\text{Var}\,\Delta W = E[(\Delta W - E\Delta W)^2]$$
$$= E\left[\left(\sum \frac{\partial f}{\partial X_i}(\Delta X_i - E\Delta X_i)\right)^2\right]$$
$$= \sum\sum \frac{\partial f}{\partial X_i}\frac{\partial f}{\partial X_j} E\left[(\Delta X_i - E\Delta X_i)(\Delta X_j - E\Delta X_j)\right]$$
$$= \sum \left(\frac{\partial f}{\partial X_i}\right)^2 \text{var}\,\Delta X_i$$
$$+ 2\sum_{i<j}\sum \frac{\partial f}{\partial X_i}\frac{\partial f}{\partial X_j} \text{Cov}(\Delta X_i, \Delta X_j)$$

Note -- $\text{Var}\,\Delta X_i = \text{Var}(X_i - X_i^*) = \text{Var}\,X_i$

Thus error variance is observable even though errors themselves are not observable.

Slope-Area Method -- M-section solution

First-Order Error Analysis

$$Q = \sqrt{\dfrac{hw}{\displaystyle\sum_{i=1}^{M} \frac{L_i}{K_i K_{i-1}} + \sum_{i=1}^{M} (1+k_i)\frac{1}{2g}\left(\frac{\alpha_i}{A_i^2} - \frac{\alpha_{i-1}}{A_{i-1}^2}\right)}}$$

where

$$K_i = A_i^{5/3} / n_i P_i^{2/3}$$

$$k_i = \begin{cases} k' & \text{if } \dfrac{\alpha_i}{A_i^2} \geq \dfrac{\alpha_{i-1}}{A_{i-1}^2} \\ -k & \text{if } \dfrac{\alpha_i}{A_i^2} < \dfrac{\alpha_{i-1}}{A_{i-1}^2} \end{cases}$$

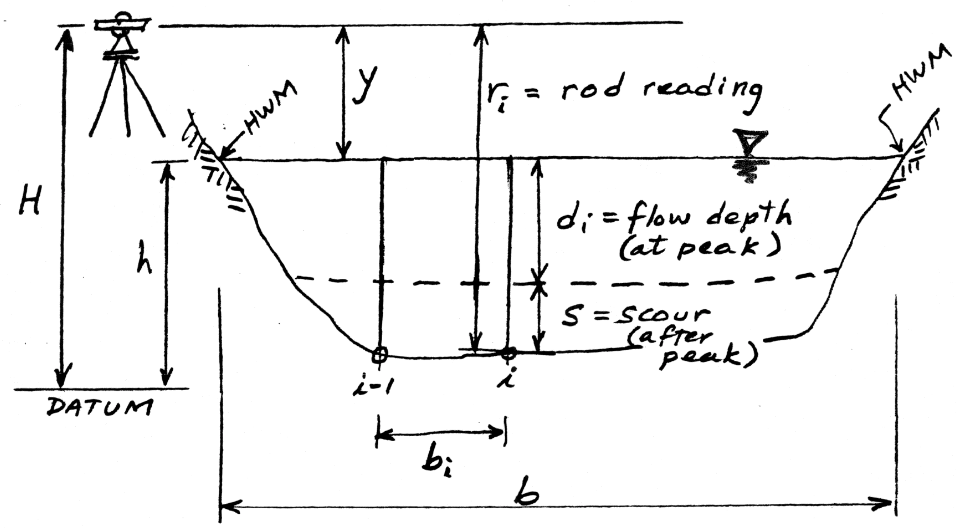
$$Q = f(h_w, k, k', \alpha_i, \underset{\underset{n_j}{\updownarrow}}{n_i}, \underset{\underset{A_j}{\updownarrow}}{A_i}, P_i, L_i)$$

These are the only covariances considered

$$\begin{aligned} Var\,\Delta Q = & \left(\frac{\partial Q}{\partial h_w}\right)^2 var\,\Delta hw \\ & + \left(\frac{\partial Q}{\partial k}\right)^2 var\,\Delta k + \left(\frac{\partial Q}{\partial k'}\right)^2 var\,\Delta k' \\ & + \sum_{i=0}^{M}\left(\frac{\partial Q}{\partial \alpha_i}\right)^2 var\,\Delta\alpha_i + \sum_{i=1}^{M}\left(\frac{\partial Q}{\partial L_i}\right)^2 var\,\Delta L_i \\ & + \sum_{i=0}^{M}\left(\frac{\partial Q}{\partial n_i}\right)^2 var\,\Delta n_i + 2\sum_{i=0}^{M-1}\sum_{j=i+1}^{M}\frac{\partial Q}{\partial n_i}\frac{\partial Q}{\partial n_j}\,Cov(\Delta n_i, \Delta n_j) \\ & + \sum_{i=0}^{M}\left(\frac{\partial Q}{\partial A_i}\right)^2 var\,\Delta A_i + 2\sum_{i=0}^{M-1}\sum_{j=i+1}^{M}\frac{\partial Q}{\partial A_i}\frac{\partial Q}{\partial A_j}\,Cov(\Delta A_i, \Delta A_j) \\ & + \sum_{i=0}^{M}\left(\frac{\partial Q}{\partial P_i}\right)^2 var\,\Delta P_i + 2\sum_{i=0}^{M}\frac{\partial Q}{\partial P_i}\frac{\partial Q}{\partial A_i}\,Cov(\Delta P_i, \Delta A_i) \end{aligned}$$

Other covariances are assumed = 0.

Cross Section Properties -- Definitions



Flow depth : $d_i = r_i - y - s$

Area : $$A = \sum_{i=1}^{N} b_i\left(\frac{d_i + d_{i-1}}{2}\right)$$
$$= \sum_{i=1}^{N} b_i\left(\frac{r_i + r_{i-1}}{2} - y - s\right)$$

Wetted Perimeter : $$P = \sum_{i=1}^{N} \sqrt{b_i^2 + (d_i - d_{i-1})^2}$$
$$= \sum_{i=1}^{N} \sqrt{b_i^2 + (r_i - r_{i-1})^2}$$

Mean Depth : $\bar{d} = A/b$

Cross Section Properties -- Error Analysis
Areas

$$A = \sum_{i=1}^{N} b_i \left(\frac{r_i + r_{i-1}}{2} - y - s \right)$$

$$\mathrm{var}\,\Delta A = \sum_{i=0}^{N} \left(\frac{\partial A}{\partial b_i}\right)^2 \mathrm{var}\,\Delta b_i + \sum_{i=0}^{N} \left(\frac{\partial A}{\partial r_i}\right)^2 \mathrm{var}\,\Delta r_i$$
$$+ \left(\frac{\partial A}{\partial y}\right)^2 \mathrm{var}\,\Delta y + \left(\frac{\partial A}{\partial s}\right)^2 \mathrm{var}\,\Delta s$$

Between cross sections j and m:

$$\mathrm{Cov}(\Delta A_j, \Delta A_m) = \frac{\partial A_j}{\partial s_j} \frac{\partial A_m}{\partial s_m} \mathrm{Cov}(\Delta s_j, \Delta s_m)$$

Partial Derivatives:

$$\frac{\partial A}{\partial b_i} = \frac{r_i + r_{i-1}}{2} - y - s \qquad \cong \bar{d}$$

$$\frac{\partial A}{\partial r_i} = \frac{b_i + b_{i+1}}{2} \qquad \cong b/N$$

$$\frac{\partial A}{\partial y} = \frac{\partial A}{\partial s} = -\sum b_i \qquad = -b$$

Errors:

$$\mathrm{var}\,\Delta b_i = \varepsilon_b^2 \cdot (b/N)^2$$
$$\mathrm{var}\,\Delta r_i = \varepsilon_r^2 \cdot \bar{d}^2$$
$$\mathrm{var}\,\Delta y = \varepsilon_h^2\, \bar{d}^2$$
$$\mathrm{var}\,\Delta s = \varepsilon_s^2 \cdot \bar{d}^2$$
$$\mathrm{Cov}(\Delta s_j, \Delta s_m) = \rho_{\Delta s} \cdot \varepsilon_s^2\, \bar{d}^2$$

Result:

$$\varepsilon_A^2 = \frac{\mathrm{var}\,\Delta A}{A^2} = \frac{\varepsilon_b^2}{N} + \frac{\varepsilon_r^2}{N} + \varepsilon_h^2 + \varepsilon_s^2$$
$$\mathrm{cov}(\Delta A_j, \Delta A_m)/A^2 = \rho_{\Delta s}\, \varepsilon_s^2$$

Error Correlation Mechanism for Manning's n and Scour

At cross sections i and j:

$$\Delta X_i = \Delta Z + \Delta Y_i$$
$$\Delta X_j = \Delta Z + \Delta Y_j$$

ΔZ is common to all cross sections -- systematic
ΔY_i are independent of each other & of ΔZ -- random

$$\text{var}\,\Delta X_i = \text{var}\,\Delta Z + \text{var}\,\Delta Y_i$$

$$\begin{aligned} \text{Cov}(\Delta X_i, \Delta X_j) &= E(\Delta X_i \cdot \Delta X_j) \\ &= E(\Delta Z^2 + \Delta Z \Delta Y_i + \Delta Z \Delta Y_j + \Delta Y_i \Delta Y_j) \\ &= \text{Var}\,\Delta Z \qquad (i \neq j) \end{aligned}$$

$$\rho_{\Delta X} = \frac{\text{var}\,\Delta Z}{\text{var}\,\Delta Z + \text{var}\,\Delta Y} = \frac{\text{systematic error}}{\text{total error}}$$

$$= \text{constant for } i \neq j$$

Summary of Error Statistics
Uniform Relative Errors

$$\frac{\mathrm{var}\,\Delta h_w}{h_w^2} = \varepsilon_{h_w}^2$$

$$\frac{\mathrm{var}\,\Delta \alpha_i}{\alpha_i^2} = \varepsilon_\alpha^2$$

$$\frac{\mathrm{var}\,\Delta L_i}{L_i^2} = \varepsilon_L^2$$

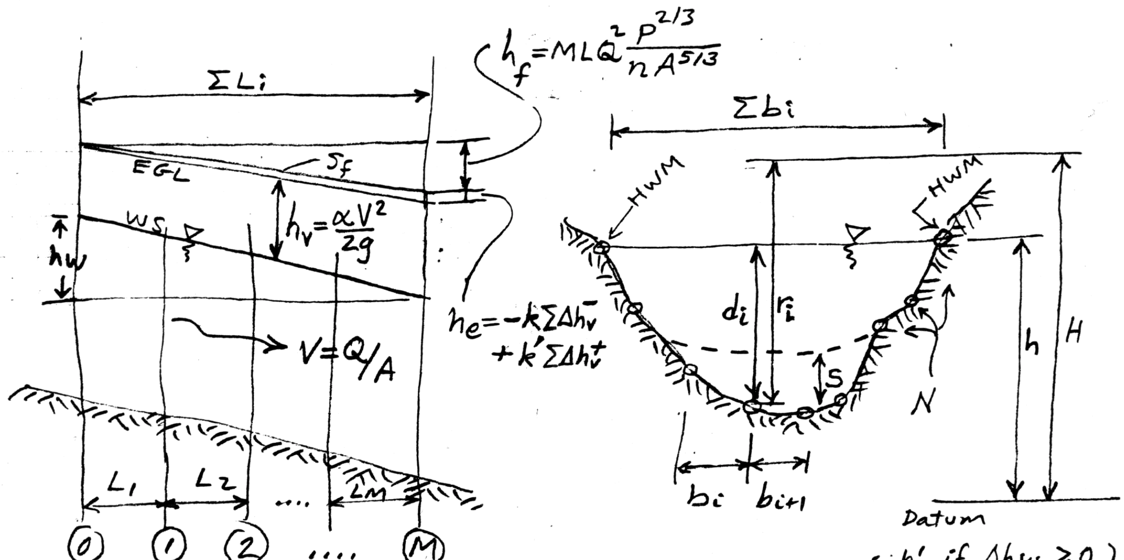
$$\frac{\mathrm{var}\,\Delta n_i}{n_i^2} = \varepsilon_n^2 \qquad \frac{\mathrm{Cov}(\Delta n_i, \Delta n_j)}{n_i\, n_j} = \rho_{\Delta n}\, \varepsilon_n^2$$

$$\frac{\mathrm{var}\,\Delta A_i}{A_i^2} = \varepsilon_A^2 = \frac{\varepsilon_b^2}{N} + \frac{\varepsilon_r^2}{N} + \varepsilon_h^2 + \varepsilon_s^2$$

$$\frac{\mathrm{Cov}(\Delta A_i, \Delta A_j)}{A_i\, A_j} = \rho_A\, \varepsilon_A^2 = \rho_{\Delta s}\, \varepsilon_s^2$$

$$\frac{\mathrm{var}\,\Delta P_i}{P_i} \cong \frac{1}{N}\varepsilon_b^2 \qquad \frac{\mathrm{Cov}(\Delta A_i, \Delta P_i)}{A_i\, P_i} \cong \frac{1}{N}\varepsilon_b^2$$

Slope-Area Method -- Variance of Q



$\varphi = h_f / h_w \qquad \psi = h_v / h_w$

$$k_i = \begin{cases} k' & \text{if } \Delta h_{v_i} \geq 0 \\ -k & \text{if } \Delta h_{v_i} < 0 \end{cases} \quad i = 1 \dots M$$
$$k_i = -1 \quad i = 0, M+1$$

$$\frac{Var\,\Delta Q}{Q^2} = \frac{1}{4}\frac{Var\,\Delta h_w}{h_w^2} + \frac{1}{4}\frac{\varphi^2}{M}\frac{Var\,\Delta L}{L^2}$$
$$+ \frac{var\,\Delta \bar{d}}{\bar{d}^2}\left[\left(\tfrac{5}{3}\varphi\right)^2 \frac{1-1/2M}{M} - \tfrac{5}{3}\varphi\psi\frac{k_M-k_1}{M} + \psi^2\sum_{i=0}^{M}(k_i-k_{i+1})^2\right]$$
$$+ \frac{var\,\Delta b}{b^2 N}\left[\varphi^2 \frac{1-1/2M}{M} - \varphi\psi\frac{k_M-k_1}{M} + \psi^2\sum_{i=0}^{M}(k_i-k_{i+1})^2\right]$$
$$+ var\,\Delta k \cdot \tfrac{1}{4}\psi^2\left(\frac{\Sigma\Delta h_{v_i}^-}{h_v}\right)^2 + Var\,\Delta k' \cdot \tfrac{1}{4}\psi^2\left(\frac{\Sigma\Delta h_{v_i}^+}{h_v}\right)^2$$
$$+ \frac{var\,\Delta\alpha}{\alpha^2}\cdot\tfrac{1}{4}\cdot\psi^2\cdot\sum_{i=0}^{M}(k_i-k_{i+1})^2$$
$$+ \frac{var\,\Delta n}{n}\varphi^2\left\{\rho_{\Delta n} + (1-\rho_{\Delta n})\cdot\frac{1-1/2M}{M}\right\}$$
$$+ \frac{var\,\Delta S}{\bar{d}^2}\left\{\rho_{\Delta s}\cdot\left(\tfrac{5}{3}\varphi\right)^2 + (1-\rho_{\Delta s})\left[\left(\tfrac{5}{3}\varphi\right)^2\frac{1-1/2M}{M} - \tfrac{5}{3}\varphi\psi\frac{k_M-k_1}{M} + \psi^2\sum_{i=0}^{M}(k_i-k_{i+1})^2\right]\right\}$$

Example -- Columbia River at The Dalles, Oregon

(Reference -- Barnes, WSP 1849, p. 30)

Given: $M = 3$

$n = 0.030$

$Q = 28.3 \times 10^3 \ m^3/s$

$\bar{A} \cong 8.35 \times 10^3 \ m^2$

$b \cong 490 \ m$

$\bar{d} \cong 17.4 \ m$

$R \cong 16.8 \ m$

$V \cong 3.35 \ m/s$

$\Sigma L_i \cong 1280 \ m$

$h_w = 0.338 \ m$

$(\alpha = 1, \ k = 1/2, \ k' = 0)$

i	A_i	k_i	Δk^2	$\frac{h_{v_i}}{h_v} = \left(\frac{\bar{A}}{A_i}\right)^2$	$\frac{\Delta h_v}{h_v}$
0	8.7	−1	1	.90	--
1	8.4	0	0	.98	.08
2	8.2	0	1/4	1.02	.04
3	8.5	$-\frac{1}{2}$	1/4	.96	−.06
--	--	−1	--	--	--

$\Sigma \Delta k^2 = 1.5$

$k_M - k_1 = -\frac{1}{2}$

$\Sigma \frac{\Delta h_v^+}{h_v} = 0.12$

$\Sigma \frac{\Delta h_v^-}{h_v} = -.06$

$\frac{\Delta h_v}{h_v} = +.06$

Compute:

$$h_v = \alpha V^2/2g \cong 1\cdot(3.35)^2/2 \times 9.81 = 0.572 \ m$$

$$h_f = h_w - \Delta h_v - h_e = 0.338 - 0.572 \times (+.06 + \tfrac{1}{2} \times .06) = 0.286$$

$$\varphi = h_f/h_w = 0.85 \qquad \psi = h_v/h_w = 1.70 \leftarrow \bigstar$$

$\cdot 10^{-2} \times .25 + 5.35$

Evaluate:

$$\frac{Var\Delta Q}{Q^2} = \frac{Var\Delta h_w}{h_w^2}\cdot\frac{1}{4} + \frac{Var\Delta\bar{d}}{\bar{d}^2}\cdot 5.35 + \frac{Var\Delta b}{b^2}\cdot\frac{4.84}{M} + \frac{Var\Delta L}{L^2}\times 0.060$$

$$+ \frac{Var\Delta\alpha}{\alpha^2}\cdot 1.10 + Var\Delta k'\cdot 0.011 + Var\Delta k\cdot 0.003$$

$$+ \frac{Var\Delta n}{n^2}\left[\rho_{\Delta n}\cdot 0.71 + (1-\rho_{\Delta n})\cdot 0.20\right]$$

$$+ \frac{Var\Delta S}{\bar{d}^2}\left[\rho_{\Delta S}\cdot 1.98 + (1-\rho_{\Delta S})\cdot 5.35\right]$$

Example -- Columbia River at The Dalles -- cont'd

Analysis of Discharge Error

x	$\frac{\sqrt{var\,\Delta x}}{x}$		C_x	$C_x \cdot \frac{Var\,\Delta x}{x^2}$
h_w	10%		0.25	25.0×10^{-4}
$\bar{d}$	2%		5.35	21.4 "
b	1%	$N=10$	0.48	.5 "
L	1%		0.06	.1 "
k'	0.10		0.011	1.1 "
k	0.10		0.003	.3 "
α	10%		1.10	110. "
n	10%	$\rho = 1/2$	0.45	45. "
S	10% $\bar{d}$	$\rho = 1/2$	3.66	366. "
Q				569.4×10^{-4} $\sqrt{\cdot} = 0.24$

Sensitivity Analysis

x	$\frac{\sqrt{var\,\Delta Q}}{Q}$
--	24%
h_w : 20%	26
α, n : 5%	21
S : 5%	17
$\rho_{\Delta S} = 0.0$	27
$\rho_{\Delta S} = 1.0$	20
$\rho_{\Delta n} = 0.0$	23